Countability axioms

1) 1st countable if countable local base at every point.

2) 2nd countable if there is a countable base.

1) Metric $\Rightarrow$ 1st countable

$$B_x = \bigcup_{r \in (0,\infty)} B_r(x)$$

2) $\mathbb{R}^+$ uniform topology is 1st countable (since metric) but not 2nd countable.
Proof of (2):

Prelim Fact: If \( X \times Y \) is 2nd countable \( \Rightarrow \) if \( A = X \) is discrete (every pt is isolated) \( \Rightarrow \) then \( A \) is countable.

Proof: \( \forall x \in A \exists \text{ some } B_x \text{ with } \)

\[
B_x \cap A - \exists x = \emptyset.
\]

So \( B_x \cap A - \exists x = \emptyset \). So \( x = x' \Rightarrow B_x \neq B_{x'} \) but \( x = x' \Rightarrow B_x \neq B_{x'} \)

\( \exists B \) but \( B \) is countable, so there are only countably many different \( B_x \)'s, so \( \text{ a countable number of } x \)'s in \( A \).
Uncountable by Cantor diagonal.

\[ B_{13} \cap B_{13} \cap B_{13} \cap B_{13} \cap B_{13} = \emptyset. \]

Thus \( p_1 x \neq y \Rightarrow 1 \) so describe

\[ x_1 = 0 \quad y_1 = 1 \quad \text{for use in next.} \]

\[ x + y \leq \bar{x} \iff A \subseteq \bar{A} \quad \text{by \#1}. \]

Proof of Claim: \( A \subseteq \bar{A} \).

Consider \( A = 30, 13, 2 \neq R \).
A subspace of a first $\aleph_1$-countable space is first $2^{\aleph_0}$-countable and the product of first $2^{\aleph_0}$-countable spaces (with product topology) is first $2^{\aleph_0}$-countable.

Proof in book

More definitions associated with countability.
**Def:**

1. $X$ is **Lindelöf** if every open cover has a countable subcover.

2. $X$ is separable if it has a countable dense set (i.e., a countable and $\overline{A} = X$).

**Proposition:**

If $X$ is second countable, then

1. Lindelöf
2. Separable

Converse true if $X$ is metric.
Ex \( \mathbb{R} \) is separable since

\( \mathbb{Q} = \mathbb{R} \) and \( \mathbb{Q} \) is countable

\( \Rightarrow \) a 2nd countable

Explicitly let

\[
\mathcal{B} = \bigcup \mathcal{B} \setminus (q) : q \in \mathbb{Q}, r \in \mathbb{Q} \setminus \mathbb{Q} \]
(1) A is a base.

(2) \( \exists \ B_n \subseteq X \) such that \( A = \bigcap B_n \).

(3) For each \( x \in A \), let \( A_x = \bigcup \{ B_n : x \in B_n \} \). This covers \( A \).

Let \( A_1 = A \). Claim: \( A_1 \subseteq \bigcup \{ B_n : \exists x \in A \} \).

Proof: So \( x \in A \) since \( \exists B_n \) is a base.

Proof: \( x \in A \) since \( \exists B_n \) is a base.
So if $B_n \subset X \in B_n \subseteq A$
but $B_n \subseteq A_n$ also so $x \in B_n \subseteq A_n$

$\Rightarrow \bigcup A_n = X$

(2) pick $\forall n, \exists x_n \in B_n$

and $D = \exists x_{n_3} \forall x \in B_n$ so $\forall B_n$

$D \cap B_n \neq \emptyset$ so dense
(1) \( \mathbb{R}^x \) is not Lindelöf, \( \mathbb{R} \) is not separable.

(2) \( \mathbb{R}^x \times \mathbb{R}^y \) is not Lindelöf.

(3) A subspace of a Lindelöf space doesn't have to be Lindelöf.
Separation Axioms

$X$ is always T₁. Stating hypotheses:

i.e. pts are closed.

**DEF**: $X$ is regular if $\forall x \in X$

- closed $B \ x \in B \ \Rightarrow$ $\exists$ open $U$ and $V$
- $x \in U$, $B \subseteq V$ and $U \cap V = \emptyset$

(2) $X$ is normal if $\forall$

- disjoint closed $A$ and $B \Rightarrow \exists$ open $U$ and $V$
- $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$

$\text{normal} \Rightarrow \text{regular} \Rightarrow \#D$
(a) $X$ is regular $\iff$ Given $X$ and open $U$ \( x \in U \Rightarrow \exists \, \text{open } \tilde{V} \text{ with } x \in \tilde{V} \text{ and } \tilde{V} \subseteq U \)

(b) $X$ is normal $\iff$

Given closed $A$ and open $U$,

$A \subseteq U$ $\Rightarrow$ $\exists$ open $V$

$A \subseteq V$ and $\overline{V} \subseteq U$

Proof: (a) previously

(b) similar.
Thm: 1. Subspace of $HD$ is $HD$.
    Product of $HD$ is $HD$ (with product).

2. (a) Subspace of reg is reg.
    (b) Product of reg is reg.

3. NOT TRUE FOR normal

Proof: (1) HW

2 (a): $y \subseteq x$, $B \subseteq y$ rel. closure.

We showed that $Cl_B(x) \cap y = B$.

And $x \notin Cl_B(x) \Rightarrow$ use reg in $x$. 
\[ \exists u, v \in X \text{ s.t. } (u, v) = 0 \]

\[ u \cap v \text{ is empty} \]

\[ \implies \forall y \text{ and } \forall y \text{ give} \]

the separation in \( Y \).

(\( \geq b \)) Proof using previous lemma.

\[ X = \prod_X \exists \text{ is reg.} \]

\[ x \in X \text{ s.t. } x \in U \text{ open in } X. \]

pick a basis \( \prod u \) with \( x \in \prod u \in U \)
For each $x$ use reg and lem.

If $V_x$ open, $x \in V_x$ and $V_x \subseteq U_x$. Let $V = \prod V_x$ open, $x \in V$, $V = \prod V_x$ and $V \subseteq \prod U_x \subseteq U$ so $V$ is regular.