Countability axioms

(1) 1st countable if countable local base at every point

(2) 2nd countable if there is a countable base.

(1) metric $\Rightarrow$ 1st countable

$\cancel{B_x}$ $\mathcal{B}_x = \{ B_r(x) : r \in [0,\infty) \cap \mathbb{Q} \}$

(2) $\mathbb{R}^{\mathbb{Z}_+}$ uniform topology is 1st countable (since metric) but not second countable.

Proof of (2):

Prelim Fact: If $(X, \mathcal{T})$ is 2nd countable $\Rightarrow$ if $A \subseteq X$ is discrete (every pt is isolated) then A is countable.

Proof $\forall x \in A$ $\exists$ some B_x with

so $B_x \cap A - \{x\} = \emptyset$.

$x = x' \Rightarrow B_x \neq B_{x'}$ but

$\mathcal{B}$ but $\mathcal{B}$ is countable, so there are only countable many different B_x's so a countable number of x's in A.

<u>consider</u> $A = \{0,1\}^{\mathbb{Z}_+} \subseteq \mathbb{R}^{\mathbb{Z}_+}$

claim: (1) it is discrete

(2) it is uncountable.

implying that $\mathbb{R}^{\mathbb{Z}_+}$ is not 2nd countable.

~~claim~~

proof $d(1)$ in A

If $\underline{x} \neq \underline{y} \Rightarrow \exists\, i$ with

$x_i = 0 \quad y_i = 1$ (or vice versa)

$\Rightarrow \bar{\rho}(\underline{x}, \underline{y}) = 1$ so discrete

$B_{1/3}(\underline{x}) \cap B_{1/3}(\underline{y}) = \emptyset$.

uncountable by Cantor diagonal.

Thm: A subspace of a first 2nd countable space is first 2nd countable and the countable product of first 2nd countable spaces (with product topology) is first 2nd countable.

$\boxed{\text{Proof in book}}$

More definitions associated with countability.

DEF:

(1) X is <u>Lindelöf</u> if every open cover has a ~~finite~~ countable subcover.

(2) X is separable if it has a countable dense set. (i.e. A countable and $\bar{A} = X$).
in cardinality

Pr'n If ~~2nd~~ X is second countable

$\Longrightarrow$ (1) Lindelöf
(2) separable

Converse true if X is metric.

Ex $\mathbb{R}_s$ is separable since

$\overline{\mathbb{Q}} = \mathbb{R}$ and $\mathbb{Q}$ is countable

$\Rightarrow$ 2nd countable

explicitly let

$$\mathcal{B} = \{ B_r(q) : q \in \mathbb{Q}, r \in [0,\infty) \cap \mathbb{Q} \}$$

(1) countable base is $\{B_n\}_{n \in \mathbb{Z}_+} = \mathcal{B}$

and $\mathcal{A}$ is an open cover.

$\forall n \; \exists A_n \in \mathcal{A}$ with $B_n \subseteq A_n$

Let $\mathcal{A}' = \{A_n\}$, claim this covers

~~Proof $x \in A$ and $\mathcal{A}$ is open $\mathcal{B}$ is a base so $\exists B_n$ with $x \in B_n \in$~~

<u>Proof</u> $x \in X$, $\exists A \in \mathcal{A}$

with $x \in A$, since $\{B_n\}$ is a base.

so $\exists$ B_n with $x \in B_n \subseteq A$

by $B_n \subseteq A_n$ also so $x \in B_n \subseteq A_n$

$\Rightarrow \bigcup A_n = X$

(2) ~~Pick x_n~~ $\forall n$, pick $x_n \in B_n$

and $D = \{x_n\}_{n \in \mathbb{Z}_+}$ so $\forall B_n$,

$D \cap B_n \neq \emptyset$ so dense $\blacksquare$

(1) $\mathbb{R}_\ell$ (lower limit topology)

is Lindelöf, 1st. countable, separable

not 2nd countable.

(2) $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is no Lindelöf.

(3) subspace of a Lindelöf space

doesn't have to be Lindelöf.

In book

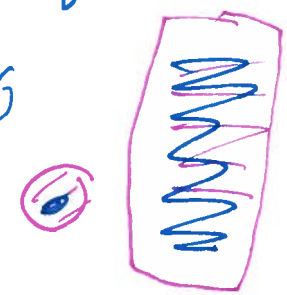
Separation Axioms

X is always T_1 ← standing hypothesis

i.e. pts are closed.

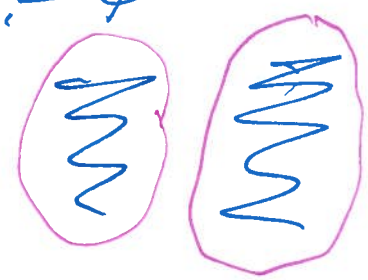
DEF: (1) X is regular if $\forall x \in X$ closed B $x \notin B \Rightarrow \exists$ open U and V

$x \in U$, $B \subseteq V$ and $U \cap V = \emptyset$



(2) X is normal if $\forall$ disjoint closed A and B $\Rightarrow \exists$ open U and V

$A \subseteq U$ $B \subseteq V$ and $U \cap V = \emptyset$

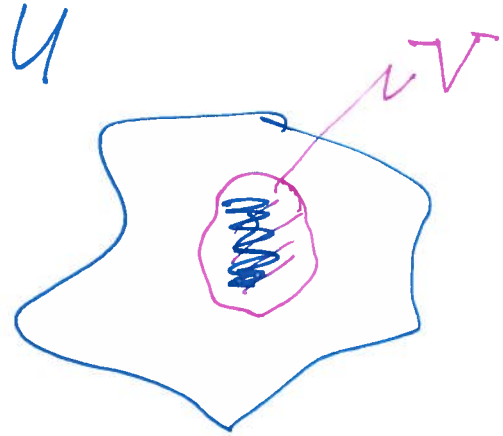


$\boxed{\text{normal} \Rightarrow \text{regular} \Rightarrow \text{HD}}$

USEFUL LEMMA

(a) X is regular $\iff$ Given x and open U $x \in U$ $\Rightarrow$ open V with $x \in V$ and $\bar{V} \subseteq U$

(b) X is normal $\iff$ given closed A and open U with $A \subseteq U$ $\Rightarrow$ $\exists$ open V with $A \subseteq V$ and $\bar{V} \subseteq U$



<u>Proof</u> (a) previously
(b) similar.

Th'm (1) Subspace of HD is HD

product of HD is HD (with prod top)

(2) (a) Subspace of reg is reg

(b) product of reg is reg.

(3) <u>NOT TRUE FOR normal</u>

Proof (1) HW

2(a): $Y \subseteq X$, $B \subseteq Y$ rel closed

$x \in Y$ $x \notin B$. We

showed that $Cl_X(B) \cap Y = B$

and $x \notin Cl_X(B) \Rightarrow$ use reg in X

$\exists\, U, V \quad x \in U \quad \mathcal{U}_{\mathcal{X}} \mid B) = V$

$U \cap V$ is empty

$\Rightarrow U \cap Y$ and $V \cap Y$ give the separation in Y.

(2 b) Proof using previous lemma.

$X = \prod X_\alpha \quad X_\alpha$ is reg.

$\underline{x} \in X \quad \underline{x} \in U$ open in X.

pick a basis $\prod U_\alpha$ with $\underline{x} \subseteq \prod U_\alpha \subseteq U$

For each α use reg and lemma.

$\exists$ V_α open, $x_\alpha \in V_\alpha$ and

$\overline{V_\alpha} \subseteq U_\alpha$. Let $V = \prod V_\alpha$

open, $\underline{x} \in V$, $\overline{V} = \prod \overline{V_\alpha}$

and $\overline{V} \subseteq \prod U_\alpha \subseteq U$ so

$\underline{X}$ is regular.