**DEF:** $X$ is normal if for all closed $A$ and $B$ they have disjoint nbhds.

Which spaces are normal?

1. Regular + 2nd countable $\Rightarrow$ normal
2. Metric $\Rightarrow$ normal
3. Cpt + HD $\Rightarrow$ normal

Big theorems about normality.
(1) **Urysohn's Lemma**: \( X \) is normal \\
A, B are disjoint and closed. \\
\[ \Rightarrow \exists \text{ a continuous function} \]
\[ f : X \rightarrow \Sigma_{011} \]
with \( f(A) = \Sigma_{0} \) \( f(B) = \Sigma_{1} \)

**Note**: This doesn't say that 
\[ f^{-1}(0) = A \quad \text{or} \quad f^{-1}(1) = B \]
That requires more hypotheses.
(2) Gruenwald's metrization theorem

Regular + 2nd countable \Rightarrow 

\( \mathcal{X} \) is metrizable i.e. there is a metric which generates the topology.

(3) Tietze Extension \( \mathcal{X} \) is normal

A \subseteq \mathcal{X} closed and \( f : A \rightarrow \{0, 1\} \)

\[ \Rightarrow \exists \text{ continuous } g : \mathcal{X} \rightarrow \{0, 1\} \]

That extends \( f \) on \( A \).

\[ g|A = f. \]
Back to proof of normality theorems

Proof: Recall Reg. \( \Rightarrow x \in \overline{W} \) open

\[ \Rightarrow \forall y \in x \text{ with } \overline{F} \subseteq W \]

let \( B \) be a countable basis?

Given \( A \) and \( B \) closed

\[ A \cap B = \emptyset \]
Start \( x \in A \) use regularity to get a \( \text{wfd} \) \( W \) with \( W \cap B = \emptyset \) then use regularity to get a \( \text{nhd} \) \( V \) of \( x \) with \( \overline{V} \subseteq W \), now choose \( U \subseteq B \) with \( x \in U \subseteq V \) and \( \overline{U} \subseteq \overline{V} \subseteq W \) so \( \overline{U} \cap B = \emptyset \) also.

Do this for each \( x \).

Since we have a countable base \( \Rightarrow \exists \) countable set \( \bigcup_{\text{base}} \Rightarrow \exists \text{countable set} \)

\[ \exists \cup_{n \in \mathbb{N}} \subseteq B, A \subseteq \cup_{n \in \mathbb{N}} \text{ and then } \forall n \in \mathbb{N}, \quad \overline{U_n} \cap B = \emptyset. \]
Do the same thing for \( B \), then \( F \) contains some \( x \geq V_n \leq B \), \( B \subseteq U V_n \) and \( \forall n, \overline{V}_n \cap A = \emptyset \).

\[
A \subseteq U U_n \quad B \subseteq V V_n
\]

but we shouldn't have \( A \cap B \neq \emptyset \).

(\( U U_n \) \( \cap V V_n \) = \( \emptyset \)).

**Trick**

\[
U_n' = U_n - \bigcup_{l=1}^{n} V_l
\]

\[
V_n' = V_n - \bigcup_{l=1}^{n} U_l
\]

These are open. \( A \subseteq U U_n' \) since each \( x \in A \) is in some \( U_n \), also \( B \subseteq V V_n' \).
and $U_{i_n'}$ and $V_{i_n'}$ are disjoint.

Why? Say $x \in U_{i_1'} V_{i_n'}$

$\Rightarrow \exists j, k \ x \in U_j' \land V_k'$, say $j \leq k$

$x \in U_j' \Rightarrow x \in U_j$

$x \in V_k' \Rightarrow x \notin U_j$

Contradiction. Similar $j \geq k$, switch roles of $U_j'$ and $V_k'$. 
Assume \( A, B \) closed, disjoint.

Recall \( d(x, A) \) is continuous in \( x \) and \( d(x, A) = 0 \iff x \in \overline{A} = A \).

Given \( a \in A \), \( d(a, B) = \varepsilon_a > 0 \).

Given \( b \in B \), \( d(b, A) = \varepsilon_b > 0 \).

Given \( b \in B \), \( \overline{B}_{\varepsilon_b}(b) \leq B_{\varepsilon_b}^c(b) \).

\( U = \bigcup_{a \in A} B_{\frac{\varepsilon_a}{2}}(a) \) and \( V = \bigcup_{b \in B} B_{\frac{\varepsilon_b}{2}}(b) \).

These are open, and \( U \cup V = \emptyset \).

Since
\[ x \in U \supset \forall \overline{V} \]

\[ \Rightarrow \exists a, b \ x \in B_{\frac{\varepsilon_a}{2}}(a) \cap B_{\frac{\varepsilon_b}{2}}(b) \]

\[ x \in B_{\frac{\varepsilon_a}{2}}(b) \]

\[ d(a, b) \leq d(a, x) + d(x, b) \leq \frac{\varepsilon_a}{2} + \frac{\varepsilon_b}{2} \leq \max(\varepsilon_a, \varepsilon_b) \]

Impossible since 

\[ a \in A \]

\[ b \in B. \]