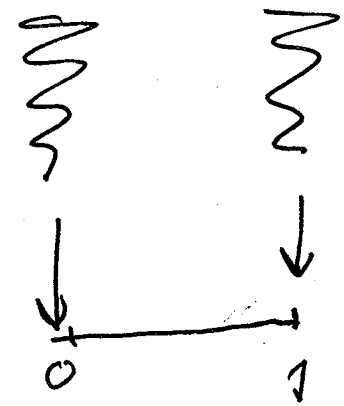
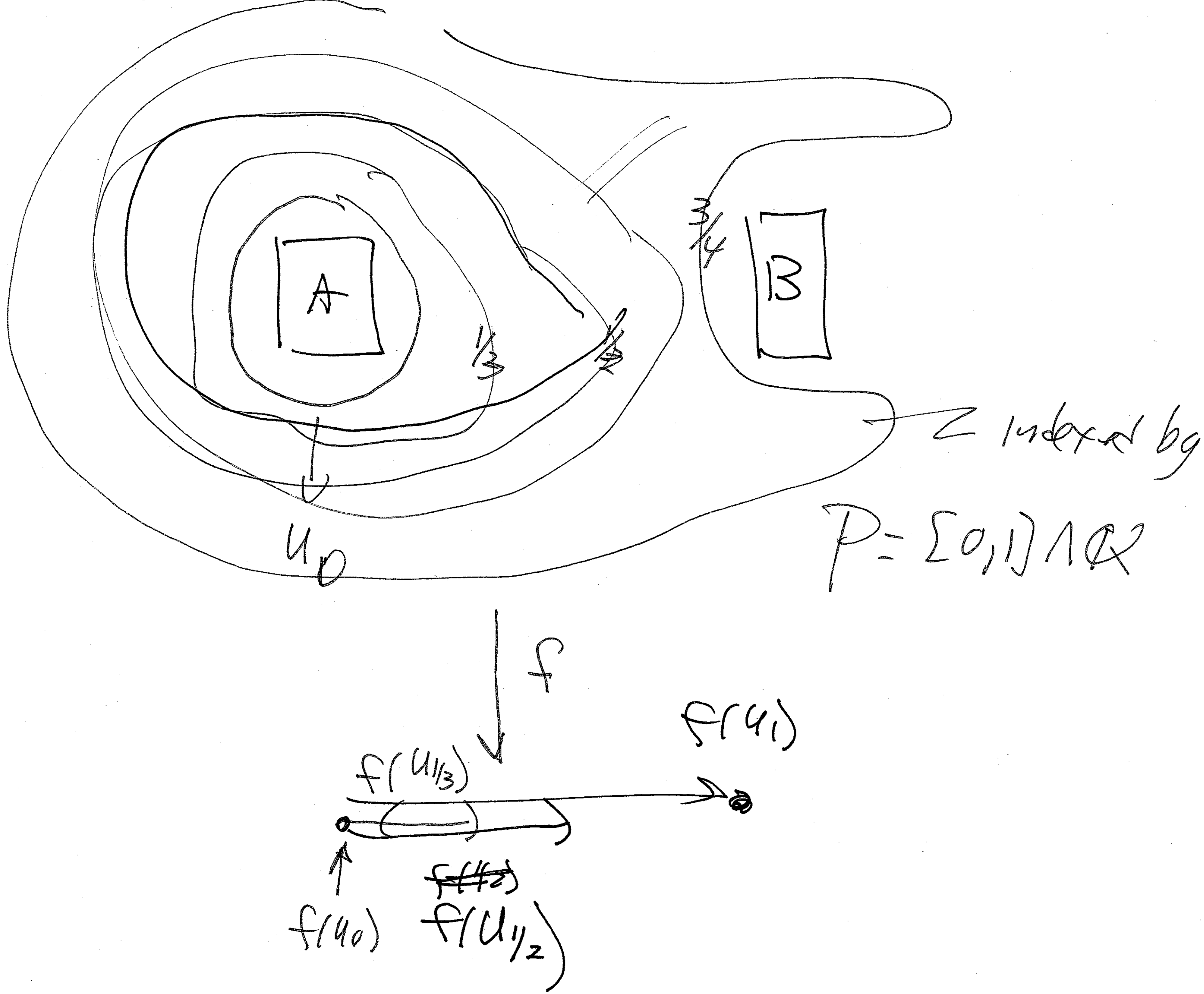
Urysohn's Lemma

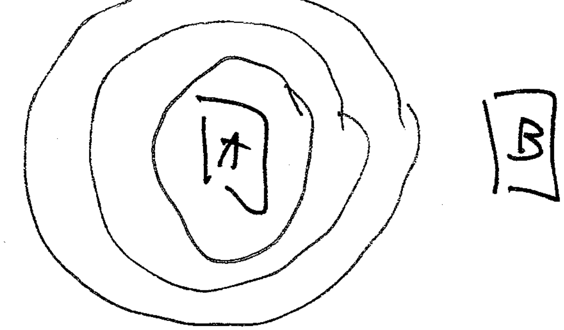
X is normal, A and B disjoint closed sets $\Rightarrow \exists\, f: X \to [0,1]$ with $f(A) = 0$, $f(B) = 1$



Curtis


A
B
1/3
1/2
3/4
u_0
$\mathcal{U}$ indexed by
$P = [0,1] \cap \mathbb{Q}$
f
$f(u_1)$
$f(U_{1/3})$
$f(u_0)$
~~f(1/2)~~
$f(U_{1/2})$

② **Step 1** claim $\exists$ family of open sets U_p with



$$A \subseteq U_0$$

$$U_1 = X - B$$

and for $q, p \in [0,1] \cap \mathbb{Q}$

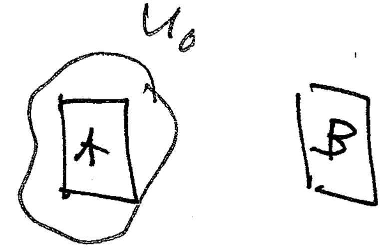
$$\boxed{p < q \Rightarrow \overline{U_p} \subseteq U_q} \quad (*)$$

Proof By induction: let $\{p_n\}_{n=0,1,\dots}$ be an enumeration of $P = [0,1] \cap \mathbb{Q}$ with $p_0 = 0$, $p_1 = 1$

eg $P = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \dots\}$.

Let $U_1 = \overline{X} - B$

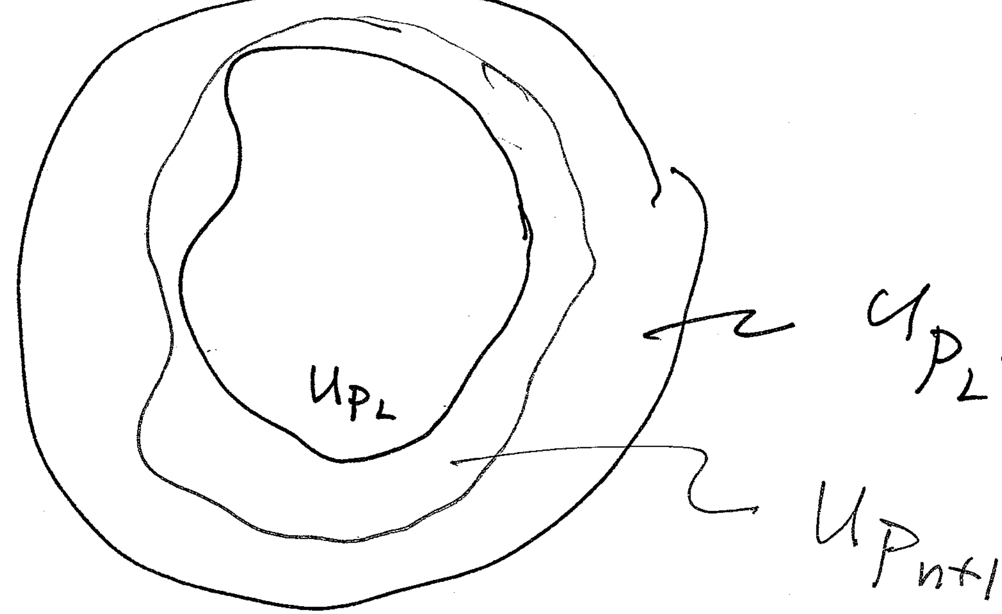
by normality $\exists\, U_0$, $A \subseteq U_0$ and $\overline{U_0} \subseteq U_1$



Assume all $p \in P_n$ $= 1^{st}$ n elements of $\overline{P}$ satisfy $(*)$

Consider $\overline{P}_{n+1} = \overline{P}_n \cup \{p_{n+1}\}$ ... $*$

So $\exists\, p_i, p_i' \in P_n$ w/ $p_i < p_{n+1} < p_i'$ and these are the immediate predecessor + successor.



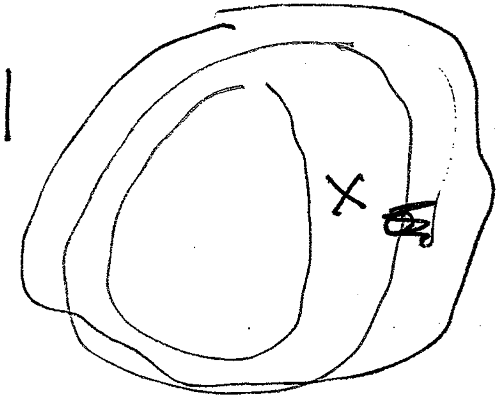
Using normality, Get $U_{p_{n+1}}$ with $\overline{U_{p_L}} \subseteq U_{p_{n+1}}$ and $\overline{U_{p_{n+1}}} \subseteq U_{p_{L'}}$

Then not hard to check that all $p, q \in I_{n+1}$ satisfy $*$

(2) Extend the family ~~by letting~~ to all $p \in \mathbb{Q}$ by

$$U_p = \emptyset \quad p < 0$$

$$U_p = X \quad p \geq 1$$



(3) Let $Q(x) = \{p : x \in U_p\}$

$\boxed{\text{Notice that}}$ $0 \leq y, \ \forall y \in Q(x)$

and ~~$Q(x) \leq 1$~~

$Q(x)$ is bounded below

$\Rightarrow$ Let $f(x) = \inf Q(x)$

Claim: $f(x)$ is the desired function

8 (4) $x \in A \Rightarrow f(x) = 0$ since $x \in U_0$.

$x \in B \Rightarrow f(x) = 1$ since $x \notin U_p$ for all $p \leq 1$

(5) $\boxed{\text{Now } f \text{ is continuous}}$

<u>Claim</u>: (1) $x \in \overline{U_r} \Rightarrow f(x) \leq r$

(2) $x \notin U_r \Rightarrow f(x) \geq r$

<u>Pf of Claim</u>: (1) $x \in \overline{U_r} \Rightarrow x \in U_s \ \forall s > r$

$\Rightarrow Q(x)$ contains all $s > r$

$\Rightarrow \inf(Q(x)) \leq r$

(2) $x \notin \mathcal{U}_r \Rightarrow x \notin \mathcal{U}_s \ \forall s < r$

so $Q(x)$ contains no rational $< r$

so $\inf(Q(x)) \geq r$. $\boxed{2}$

For continuity Pick $x_0 \in X$

and $(c,d) \subseteq \mathbb{R}$ with $f(x_0) \in (c,d)$

and we find nbd $\mathcal{U}$ of x with

$f(\mathcal{U}) \subseteq (c,d)$. choose

$c < p < f(x_0) < q < d$, $p, q \in \mathbb{Q}$

Let $\mathcal{U} = \mathcal{U}_q - \overline{\mathcal{U}_p}$

$U = U_q - \overline{U_p}$

Claim: $f(U) \subseteq (c,d)$.

1st show that $x_0 \in U$.

Since $f(x_0) < q \Rightarrow$ by contrapositive of claim 2 $x_0 \in U_q$.

and $f(x_0) > p \Rightarrow$ contrapositive of 1, $x_0 \notin \overline{U_p}$

Now pick arbitrary $x \in U$.

So $x \in U_q \subseteq \overline{U_q}$ by claim 1

$f(x) \leq q$.

and $x \notin \overline{U_p} \Rightarrow x \notin U_p$ by

Claim 2, $f(x) \geq p$

So $f(x) \subseteq [p, q] \subseteq (c, d)$

as required. $\blacksquare$

__DEF__: If $A, B \subseteq \underline{X}$ and $\exists f: \underline{X} \to [0,1]$ with $f(A) = \{0\}$, $f(B) = \{1\}$ $\Rightarrow$ A and B are said to be separated by a continuous function.

$\boxed{\text{Urysohn Lemma}}$ If every pair of disjoint closed sets can be separated by disjoint open ~~of~~ sets $\Rightarrow$ they can be separated by a continuous function.

DEF: X is called <u>completely regular</u> if it is T_1 and for all x_0 closed A, $x_0 \notin A$

x_0 and A can be separated by a cont. funct.

FACT: reg $\supsetneq$ compreg $\supsetneq$ normality.

FACTS

(1) reg x reg is reg.
subspace of reg is reg

(2) Same for complety reg.

(3) Not true for normal.

(a) norm x norm $\neq$ norm is hard

(b) Subspace of normal not normal
remove arc from a squar