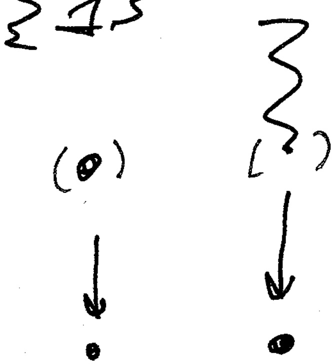
Counterexamples in Topology

Completly reg $\Rightarrow$ reg.

Given: $x_0 \notin B$, B closed say there exists cont $f : X \to [0,1]$ with

$f(x_0) = \{0\}$ and $f(B) = \{1\}$



$\Rightarrow$ $f^{-1}([0, 1/3))$ and

$f^{-1}((2/3, 1])$ are the

desired open set

Urysohn metrization Th'm

Reg. + 2nd countable $\Rightarrow$ metrizable

Ideas in proof:

(1) Use Urysohn lemma to find
$\exists$ a collect $\{f_n\}$, $f_n : X \to [0,1]$
$n \in \mathbb{Z}_+$ with the property $\forall x_0 \in X$,
$\forall U$ nbhds of x_0 $\exists$ some n with

$$f_n(x_0) > 0$$
$$f_n(X - U) = 0$$

(2) Define $F : X \to [0,1]^{\mathbb{Z}_+}$ product top.

via $F(x) = (f_1(x), f_2(x), \ldots)$

$\Longrightarrow$ F is a homeomorphism onto its image so $F(X) \subseteq [0,1]^{\mathbb{Z}_+}$ a metrizable space, so $F(X)$ is metrizable and Thus so is X.

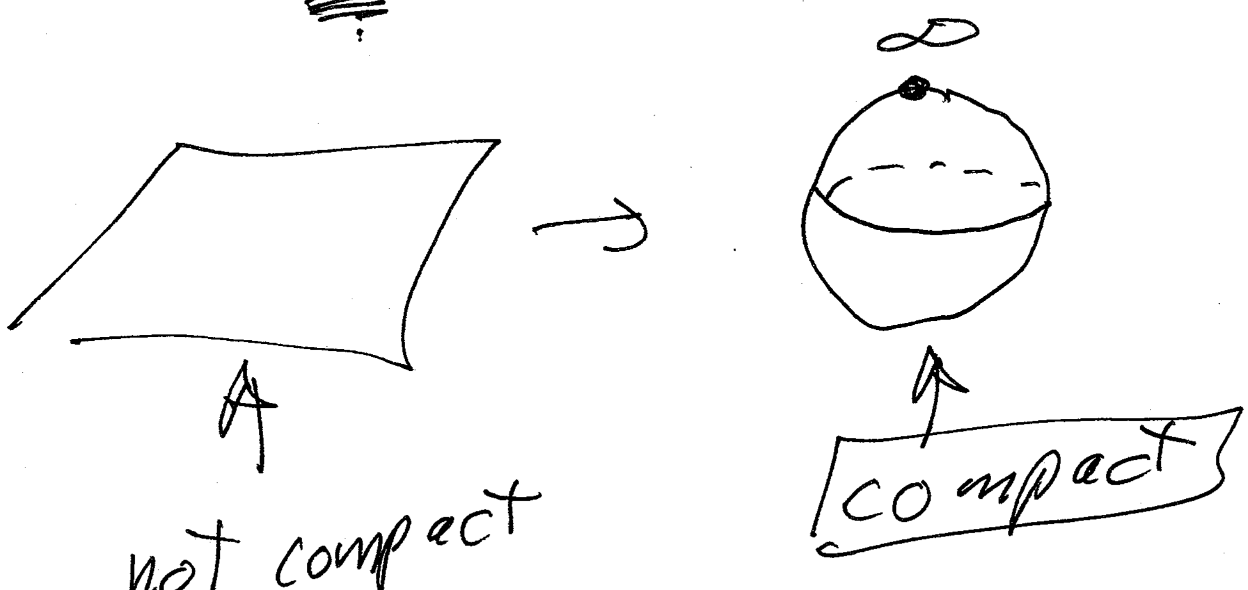
Tietze extension Theorem

X normal, $A \subseteq X$ closed, any $f : A \to [0,1]$ cont can be extended to cont $g : X \to [0,1]$, $g|_A = f$.

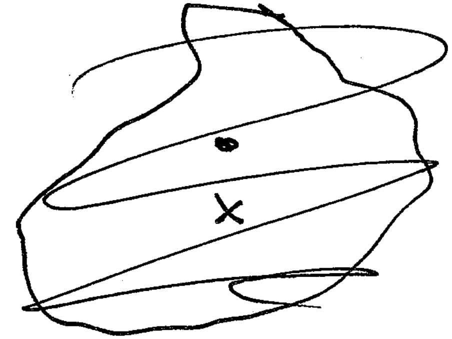
One point compectification

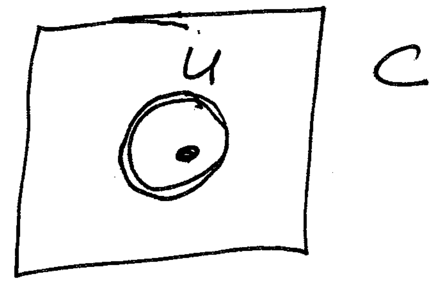
eg// $\cancel{\mathbb{R}}$

$\mathbb{R} \cup \{\infty\} =$ circle (with correct topology).



DEF: X is locally cpt if $\forall x \in X$ $\exists$ cpt C and an open U with $\cancel{x \in C \subseteq U}$
$x \in U \subseteq C$





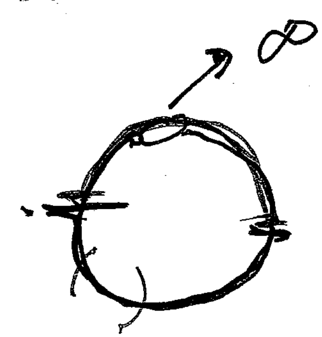
HD

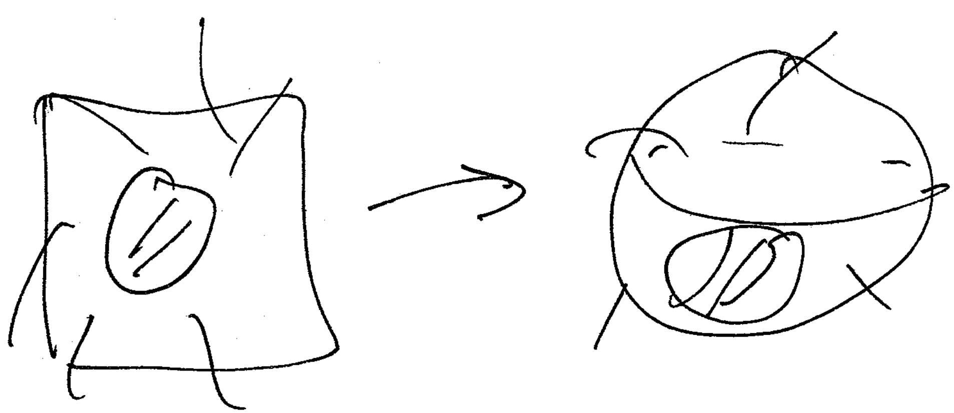
Theorem X is locally cpt
$\Longleftrightarrow$ it has a one point compactification
i.e. $\exists$ cpt, HD Y with $X \subseteq Y$
and $Y = X$ union a single point
called "∞".

Proof: $\mathcal{Y} = X \cup \{\infty\}$ with ∞ a point not in ~~$\mathcal{Y}$~~ X. Topology on $\mathcal{Y}$ has open sets:

(1) U open in X

(2) $U = \mathcal{Y} - C$ with C cpt in X





Proof: (a) This is a topology

(b) $\mathcal{Y}$ is cpt.

$\mathcal{A}$ is a cover of $\mathcal{Y}$

$\Rightarrow$ so $\exists$ a set of the type $\mathcal{Y} - C$

Let $\mathcal{A}' = \mathcal{A} - \{\mathcal{Y} - C\}$, is a cover of C $\Rightarrow$ cptness finite subcover and then this with $\mathcal{Y} - C$ is a finite subcover of $\mathcal{Y}$.

(c) $\mathcal{Y}$ is HD.

Assume $x \neq y$ in $\mathcal{Y}$

if $x, y \in \mathbb{X}$ use HD of $\mathbb{X}$.

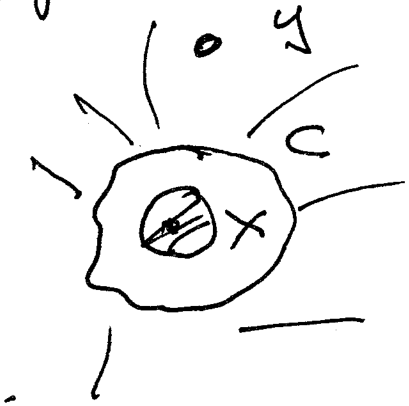
if $x \in \mathbb{X}$ $y = \infty$, then by

loc cptness $\exists$ cpt C and open

U with $x \in U \subseteq C \Rightarrow$

U and $\mathcal{Y} - C$ are the

two disjoint open sets.



$\blacksquare$

Remarks (1) converse is true

(2) If X is not compact to start then in Y $\overline{X} = Y$ so X is dense in Y

(3) one point compactification is unique up to homeomorphism.

"Any metric spaces can be completed"

eg// $\mathbb{Q}$ is a metric space, its completion is $\mathbb{R}$.

DEF: $f : (X, d) \to (X', d')$ is an isometry if

$$d(f(x), f(y)) = d(x, y) \quad \forall x, y \in X$$

$\Rightarrow$ (X, d) and (X', d') are said to be isometric, or indistinguishable as metric spaces.

when f is onto (isometry easily implies injective)

Th'm If (X, d) is metric $\Rightarrow$

$\exists$ an isometry of X to $X' \subseteq Y$

with Y complete metric

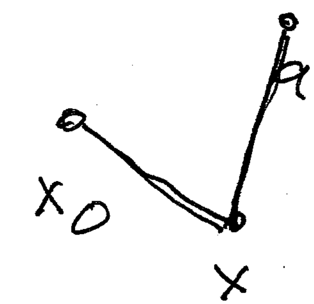
Remark: $\cancel{X'}$ $\overline{X'}$ is called the

completion of X and is unique

up to isometry.

<u>Idea</u>: $Y = \mathcal{B}(X, \mathbb{R})$ which is a complete metric space with metric

$$\rho(f,g) = \sup_{x \in X} (f(x), g(x))$$



Fix $x_0 \in X$

given $a \in X$ let

$$\phi_a(x) = d(x,a) - d(x,x_0)$$

Claim $\phi_a \in \mathcal{B}(X, \mathbb{R})$

Note that

$$d(x,a) \leq d(x,b) + d(a,b)$$

$$d(x,b) \leq d(x,a) + d(a,b)$$

$$\Rightarrow |d(x,a) - d(x,b)| \leq d(a,b) \quad (*)$$

Let $b = x_0 \Rightarrow |\phi_a(x)| \leq d(a,x_0)$

so ϕ_a is Bded. so $\phi_a \in \mathcal{B}(X,\mathbb{R})$.

now define $\Phi : X \to \mathcal{B}(X,\mathbb{R})$

via $\quad a \mapsto \phi_a$

claim this is an isometry.

or $\rho(\phi_a,\phi_b) = d(a,b)$

$$\rho(\phi_a, \phi_b) = \sup_{x \in X} \left| \phi_a(x) - \phi_b(x) \right|$$

$$= \sup \left| d(x,a) - d(x,b) \right|$$

$$\leq d(a,b) \quad \text{by } (*).$$

The sup is ~~achiet~~ achived ~~which~~ as $d(a,b)$ with $x = a$

$$\cancel{\rho(\phi_a,}\quad \left| d(a,a) - d(a,b) \right|$$

$$= \left| d(a,b) \right|$$

$$\Rightarrow \rho(\phi_a, \phi_b) = d(a,b).$$