DEF $\mathcal{S}$ is a subbase if

$$\bigcup_{S \in \mathcal{S}} S = X$$ (also called a cover)

$\boxed{\text{example } \{[a,\infty) : a \in \mathbb{R}\} \text{ is a subbase.}}$

let $\mathcal{B}$ = all finite intersections of elements of $\mathcal{S}$.

→ easy: this is a base.

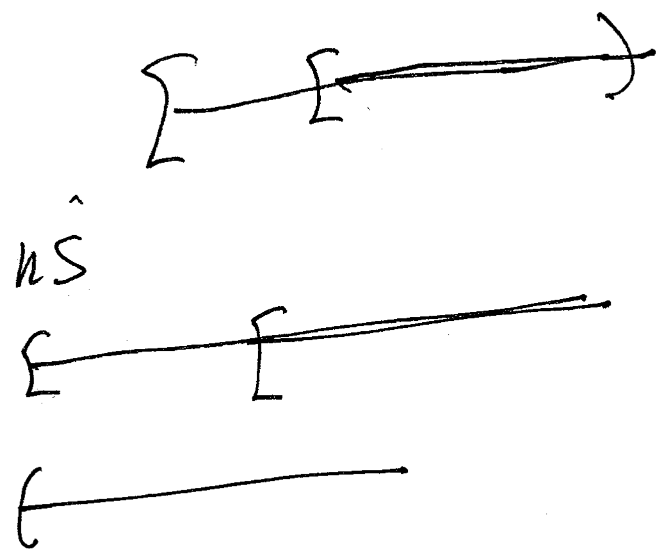
let $\mathcal{T}$ = all arbitrary unions of elements of $\mathcal{B}$

→ easy: this is topology

Back to example

$$\mathcal{S} = \{ [a, \infty) : a \in \mathbb{R} \}.$$

$$\mathcal{B} = \mathcal{S}$$



$\mathcal{T}$ = all arb unions

$$(a, \infty) = \bigcup_{n=1}^{\infty} [a + 1/n, \infty)$$

- Order topology on strictly linear ordered sets.

Let $\mathcal{B}$ be all sets of the form.

(1) (a,b)

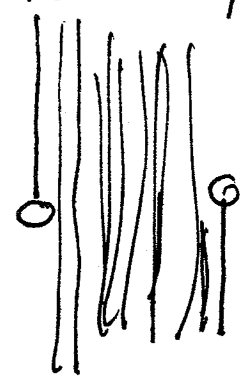
(2) $[a_0,b)$ <u>if</u> a_0 is the smallest element <u>if it exists</u>

(3) $(a,b_0]$ if b_0 is the largest element if it exists.

<u>FACT</u>: This is a <u>base</u> that generates the order topology.

Examples (1) $(\mathbb{R}, <)$ yields standard topology with base $\{(a,b)\}$

(2) Dictionary order on $\mathbb{R}^2$



$\leftarrow ((a,b),(c,d))$ with $a<c$

$((a,b),(a,c))$

$b<c$

Not the standard topology.

(3) $\mathbb{Z}_+ = \{1, 2, 3, \ldots\}$

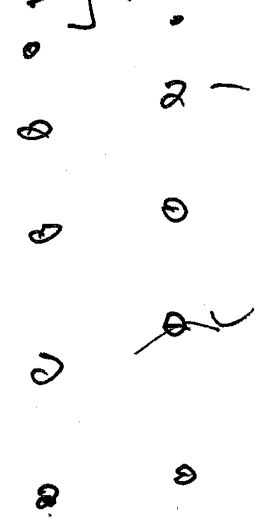
has smalest element 1.

$$[1, 2) = \{1\}$$

$$(n-1, n+1) = \{n\}$$

discrete topolgy.

(4) $\{1,2\} \times \mathbb{Z}_+$ with the dictionary topology.



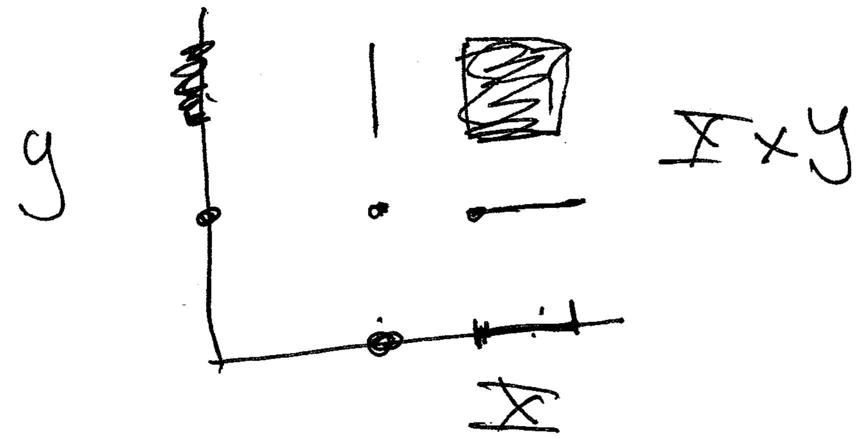
$(1,1)$ smallest

every pt is open

<u>but</u> $(2,1)$

any open interval containing $(2,1)$ contains a point $< (2,1)$ or $(1,b)$ for some b and thus $(1,b')$ for all $b' > b$ so infinitely points.

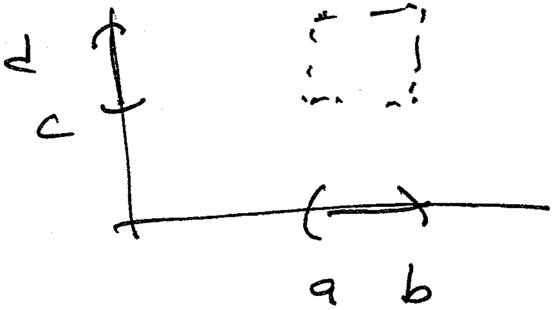
Product topology



Given $(\mathcal{X}, \mathcal{T}_{\mathcal{X}})$ and ~~$\mathcal{Y}$~~ $(\mathcal{Y}, \mathcal{T}_{\mathcal{Y}})$ $\Rightarrow$ the topology on the product $\mathcal{X}\times\mathcal{Y}$ is $\mathcal{T}_{\mathcal{X}\times\mathcal{Y}}$ which is generated by the base

$$\{U\times V : U\in\mathcal{T}_{\mathcal{X}},\ V\in\mathcal{T}_{\mathcal{Y}}\}$$

eg) In $\mathbb{R}_S \times \mathbb{R}_S$



$[a,b) \times [c,d)$ is typical base element.

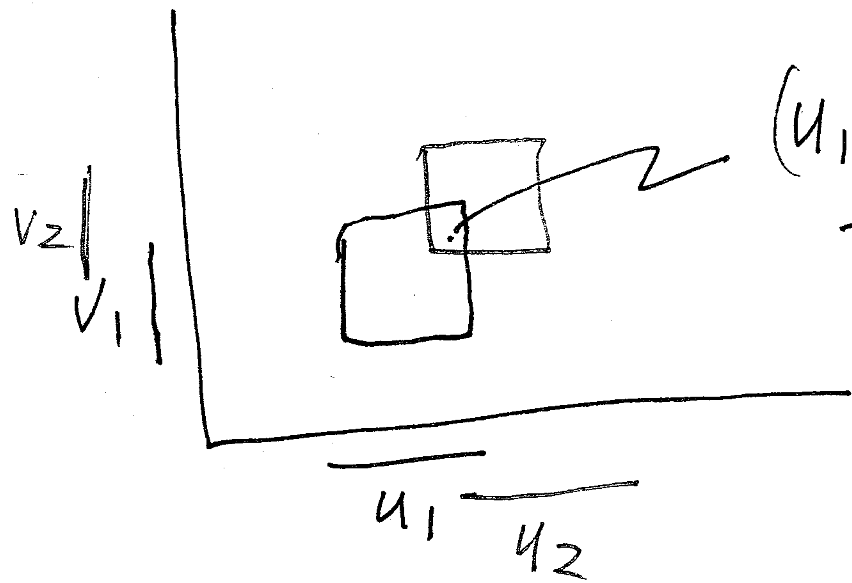
Check $\mathcal{B}$ is a base

(1) Trivial X, Y are open in $\mathcal{T}_X$ and $\mathcal{T}_Y$

(2) $(U_1 \times V_1) \cap (U_2 \times V_2)$

$= (U_1 \cap U_2) \times (V_1 \cap V_2)$

Intersection of base elements yield a base element.



$(U_1 \times V_1) \cap (U_2 \times V_2)$
$= (U_1 \cap U_2) \times (V_1 \cap V_2)$

$\mathcal{B}$ is not a topology i.e. the only open sets in $\mathcal{T}_{X \times Y}$ are not just the $U \times V$

eg!! $(U_1 \times V_1) \cup (U_2 \times V_2) \in \mathcal{T}_{X \times Y}$
but is not a product

[Now reduce to bases in $\mathfrak{X}$ and $\mathcal{Y}$]

Theorem: If $\mathcal{B}$ is a base for $\mathcal{T}_{\mathfrak{X}}$ and $\mathcal{C}$ is a base for $\mathcal{T}_{\mathcal{Y}}$ $\Rightarrow$

$$\mathcal{B}_{\mathfrak{X}\times\mathcal{Y}} = \{ B \times C : B \in \mathcal{B}, C \in \mathcal{C} \}$$

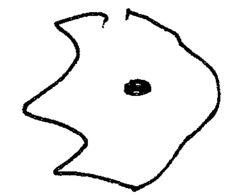
is a base for $\mathcal{T}_{\mathfrak{X}\times\mathcal{Y}}$.

Uses lemma: If & $\mathcal{T}$ is a topology on $\mathfrak{X}$ and $\mathcal{C} \subseteq \cancel{\mathcal{P}(\mathfrak{X})}\,\mathcal{T}$ has the property that for $U \in \mathcal{T}$ and and $x \in U$ $\exists C \in \mathcal{C}$ with $x \in C \subseteq U$ $\Rightarrow$ $\mathcal{C}$ is a base for $\mathcal{T}$.

Proof of Theorem: We show

$\mathcal{B}_{X \times Y}$ satisfies the conditions for $\mathcal{C}$

in the lemma. Say W is open



in $X \times Y$ and $(x,y) \in W$. So by def of the the product

topology base $\exists\, U \in \mathcal{T}_X,\ V \in \mathcal{T}_Y$

with $(x,y) \in U \times V \subseteq W$.

But $\mathcal{T}_X$ has base $\mathcal{B}$, so $\exists\, B \in \mathcal{B}$ with

$x \in B \subseteq U$ and similarly, $\exists\, C \in \mathcal{C}$ with

$y \in C \subseteq V \Rightarrow (x,y) \in B \times C \subseteq W$

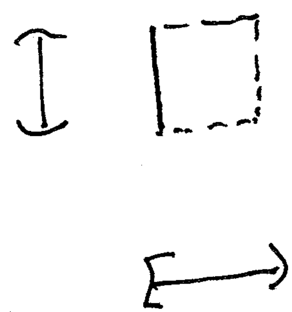
so $\mathcal{B}_{X \times Y}$ satisfies the lemma.

__Example__ $\mathbb{R}_\ell$ has base $\{[a,b)\}$

$\mathbb{R}_s$ has base $\{(a,b)\}$

base for $\mathbb{R}_\ell \times \mathbb{R}_s$ on $\mathbb{R}^2$ is

$\{[a,b) \times (c,d)\}$



But $\mathbb{R}_s \times \mathbb{R}_s$ yields $\mathbb{R}_s^2$

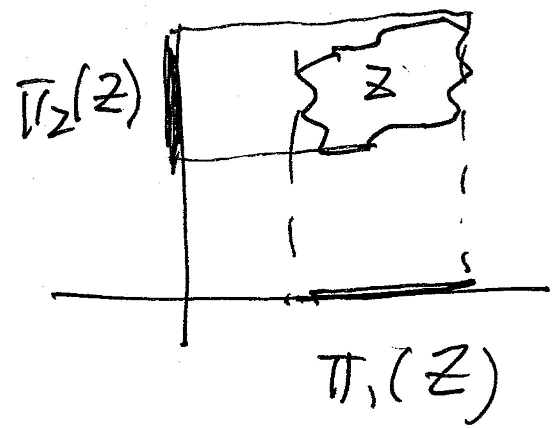
Projections

$\pi_1 : X \times Y \to X$

$\pi_1(x,y) = x$

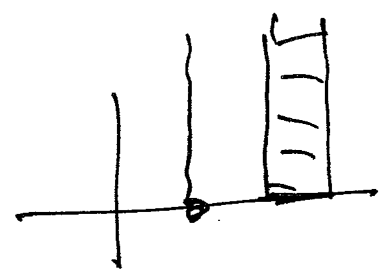
$\pi_2 : X \times Y \to Y$

$\pi_2(x,y) = y$



$\pi_1^{-1}(A) = A \times Y$

$\pi_2(B) = \cancel{B}\, X \times B$



Thus, if $U \subseteq \mathcal{J}_X \Rightarrow \pi_1^{-1}(U)$ is open in $\mathcal{J}_{X \times Y}$ $\quad \pi_1^{-1}(U) = U \times Y$

Similarly $V \subseteq \mathcal{J}_X$, $\pi_2^{-1}(V)$ open $\mathcal{J}_{X \times Y}$.

Theorem: $\{\pi_1^{-1}(U) : U \in \mathcal{J}_X\} \cup \{\pi_2^{-1}(V) : V \in \mathcal{J}_Y\}$ is a subbase for $\mathcal{J}_{X \times Y}$.

Idea in proof

$$\pi_1^{-1}(U) \cap \pi_2^{-1}(V)$$

$$= (U \times Y) \cap (X \times V)$$

$$= U \times V$$ a base element for

$$\mathcal{T}_{X \times Y}.$$