Setup

4/19/19

$$\frac{dx}{dt} = f(x,y)$$

$$\frac{dy}{dt} = g(x,y)$$

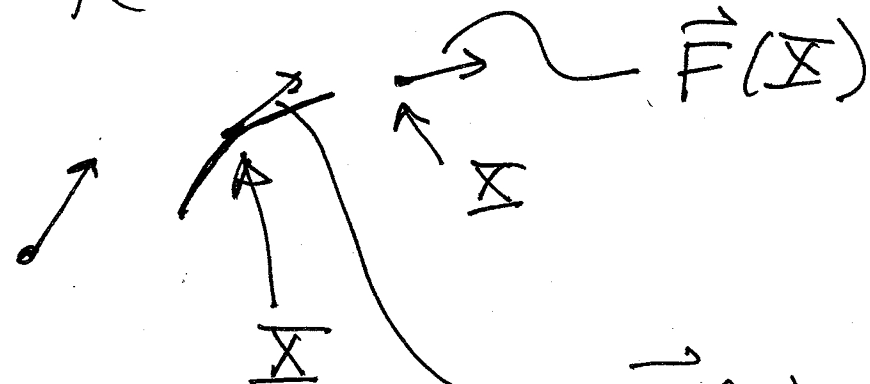
- 2 dimensional autonomous (no t in RHS)
- pictures of solutions.

$$\underline{X} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \vec{F} = (f, g)$$

$$\frac{d\underline{X}}{dt} = \vec{F}(\underline{X})$$

vector field
or velocity field

A trajectory or orbit or solution curve is a curve $\vec{\gamma}(t)$ with the velocity or tangent to $\vec{\gamma}(t)$ at each point equal to the vector field



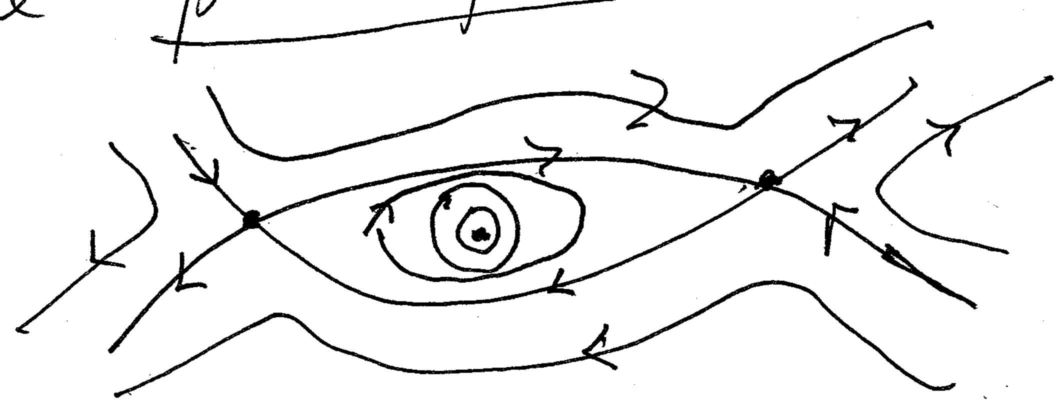
$$\vec{F}(\underline{X}) = \frac{d\gamma}{dt}(t)$$

$$\frac{d\vec{\gamma}}{dt}(t) = \vec{F}(\gamma(t))$$

$\gamma(t) = (x(t), y(t))$

$$\boxed{\begin{aligned} \frac{dx(t)}{dt} &= f(x(t), y(t)) \\ \frac{dy(t)}{dt} &= g(x(t), y(t)) \end{aligned}}$$ DE

The collection of all trajectories is the <u>phase portrait</u> of the DE

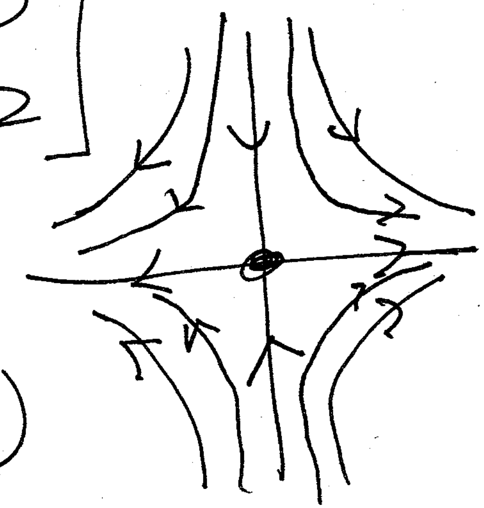


nonlinear pendulum.

$\boxed{\text{LAST time}}$

$$\frac{dx}{dt} = 2x \qquad \frac{d\underline{X}}{dt} = A\underline{X}$$

$$\frac{dy}{dt} = -2y \qquad A = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

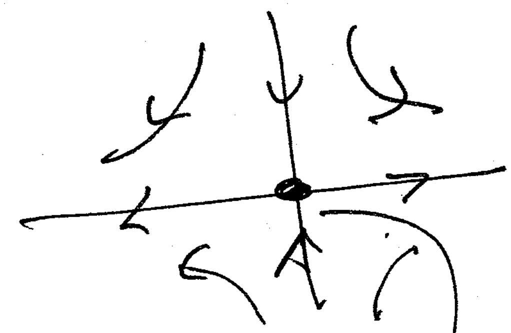


$$\vec{F}(x,y) = (2x, -2y)$$

$$\vec{\gamma}(t) = (x(0)e^{2t}, y(0)e^{-2t})$$

lie on curves $\quad xy = x(0)y(0)$

(x_0, y_0) is an equilibrium or fixed point if



$$\vec{F}(x_0, y_0)$$

i.e. trajectory through the point stays fixed pt

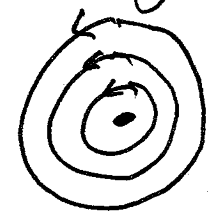
In example only fixed pt is

$(x_0, y_0) = (0, 0)$ since $\begin{aligned} 2x &= 0 \\ -2y &= 0 \end{aligned}$

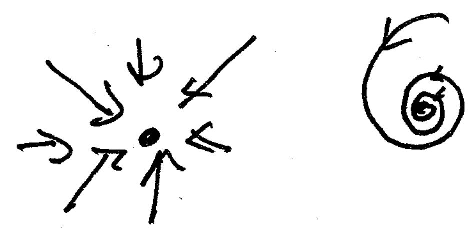
has just one soln.

equilibrium is

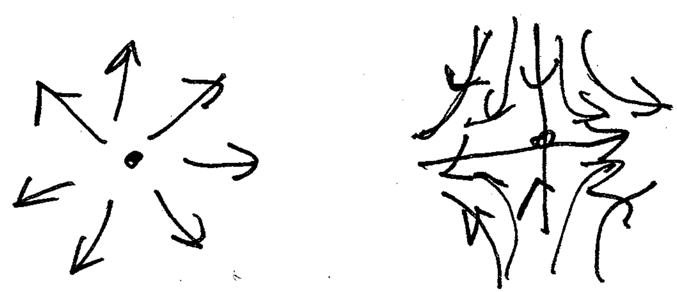
(1) <u>stable</u> if nearby trajectories stay nearby



(2) <u>asymptotically stable</u> if nearby trajectories converge to the equilibrium.



(3) <u>unstable</u> if not stable



Given a physical system or DE. 1st step is finding equilibrium

2nd step, is to classify the stability of equilibrium.

⋮

For 2nd step, the first ~~a~~ thing is the linear case ⇒ reduce to linear case.

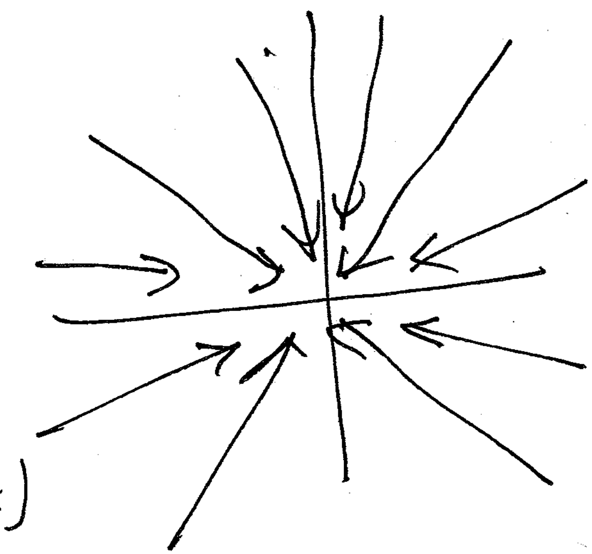
2 more examples

$$\frac{d\overline{X}}{dt} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \overline{X}$$

$$x(t) = x(0)\, e^{-2t}$$
$$y(t) = y(0)\, e^{-2t}$$

$$e^{-2t} = \frac{x(t)}{x(0)} = \frac{y(t)}{y(0)}$$

$$\boxed{y(t) = \frac{y(0)}{x(0)}\, x(t)}$$

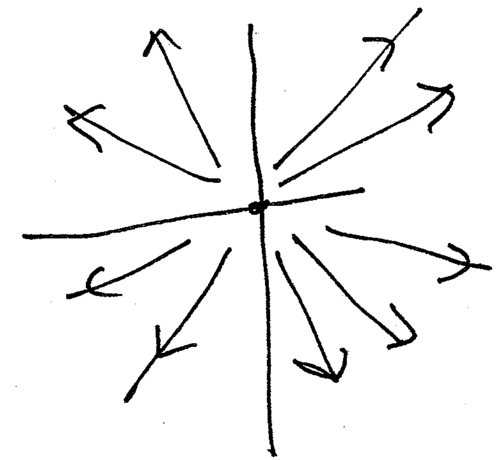


- asymp stable.
- Sink
- attractor.

$$\frac{d\mathbf{X}}{dt} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{X}$$

$$x(t) = x(0)\, e^{2t}$$

$$y(t) = y(0)\, e^{2t}$$



$$y(t) = \frac{y(0)}{x(0)}\, x(t)$$

- unstable
- source
- repeller.

Now Do general cases - stability depends (almost always) just on the eigenvalues of A in $\frac{d\overline{x}}{dt} = A\overline{x}$

Assume $\lambda_1 \neq \lambda_2$ (the eigenvalues of A) **Real** with eigenvectors $\vec{u}_1$ and $\vec{u}_2$

$$\vec{X}(t) = c_1 e^{\lambda_1 t} \vec{u}_1 + c_2 e^{\lambda_2 t} \vec{u}_2$$

is general soln.

1. $\lambda_1 < \lambda_2 < 0 \Rightarrow e^{\lambda_1 t} \to 0$
$e^{\lambda_2 t} \to 0$ so $\vec{x}(t) \to \vec{0}$ the equilibrium point to $\frac{d\vec{X}}{dt} = A\vec{X}$, since $A\vec{0} = \vec{0}$.
for all c_1 and c_2 $\boxed{\text{asymptotically stable}}$

2. $0 < \lambda_1 < \lambda_2$, $e^{\lambda_1 t} \to \infty$ $e^{\lambda_2 t} \to \infty$
as $t \to \infty$ for all c_1 and c_2
~~as~~ $\boxed{\text{unstable}}$

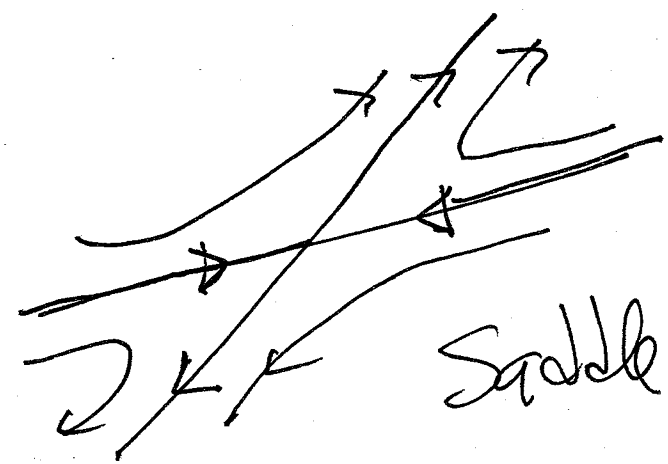
(3) $\lambda_1 < 0 < \lambda_2$ as $t \to \infty$

$e^{\lambda_1 t} \to 0 \quad e^{\lambda_2 t} \to \infty$

<u>so</u> $c_1 = 0 \quad c_2 \neq 0 \quad \vec{x}(t) \to \infty$

no matter how small c_2 is

$\boxed{\text{Unstable.}}$



Saddle point

<u>Examples</u> - Determine the stability of $\vec{0}$ as an equilibrium point to

$$\frac{d\mathbf{X}}{dt} = A\mathbf{X}$$ for each A.

(1) $A = \begin{bmatrix} -5 & 3 \\ -6 & 4 \end{bmatrix}$, $\begin{vmatrix} -5-\lambda & 3 \\ -6 & 4-\lambda \end{vmatrix} = (-5-\lambda)(4-\lambda) + 18$

$= \lambda^2 + \lambda - 2 = (\lambda+2)(\lambda-1)$

$\lambda = 1 \quad \lambda = -2$

$\boxed{\text{unstable saddle point}}$

(2) $A = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}$

$(-3-\lambda)(-\lambda)+2$

$\lambda^2+3\lambda+2=(\lambda+2)(\lambda+1)$

$\lambda=-1 \quad \lambda=-2$. stable asymp.

sink

attractor.

(3)

$$A = \begin{bmatrix} -1 & 2 \\ -4 & 5 \end{bmatrix}$$

$$(-1-\lambda)(5-\lambda)+8$$

$$\lambda^2-4\lambda+3 = (\lambda-3)(\lambda-1).$$

$\lambda = 1 \quad \lambda = 3$ both post

$\Rightarrow$ unstable, source, repellor.

Two remaining cases (1) complex eigenvalues
(2) repeated eigenvalues.