12.2 CONT

Stability of equilibrium

4/12/19

- Linear systems
- non linear equilibrium
- general systems.

CASES In Linear Systems.

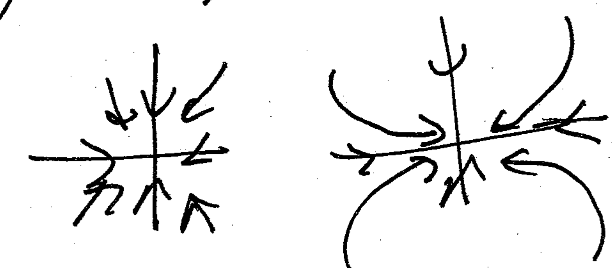
Last time

$\underline{X} = \begin{pmatrix} x \\ y \end{pmatrix}$

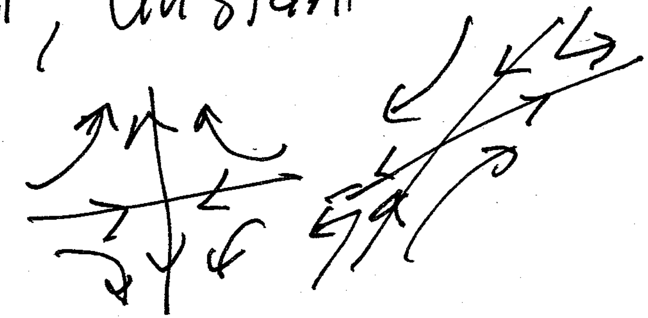
$$\frac{d\underline{X}}{dt} = A\underline{X}$$

eigenvalues $\lambda_1 \neq \lambda_2$ real nonzero

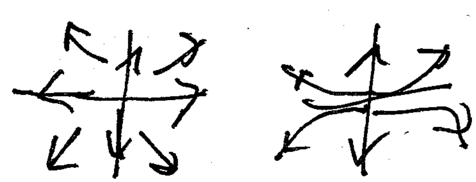
(1) $\lambda_1 < \lambda_2 < 0 \Rightarrow$ asymp stable, sink, attractor



(2) $\lambda_1 < 0 < \lambda_2 \Rightarrow$ saddle point, unstable



(3) $0 < \lambda_1 < \lambda_2 \Rightarrow$ unstable, repeller, source

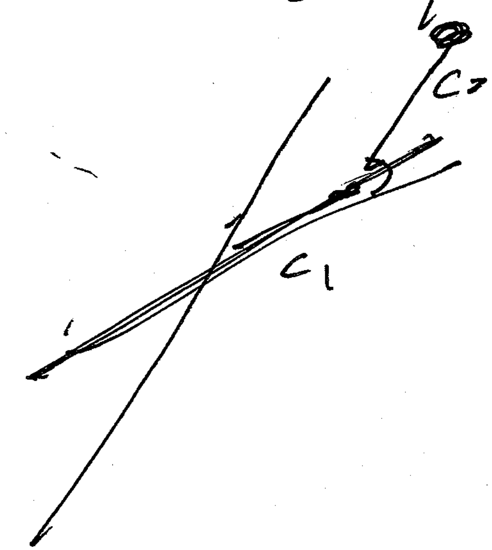


Form of the solution with e. vect $\vec{u}_1$ and $\vec{u}_2$ [2

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{u}_1 + c_2 e^{\lambda_2 t}\vec{u}_2$$

$$\vec{x}(0) = c_1 \vec{u}_1 + c_2 \vec{u}_2$$

Coordinates of $\vec{x}(0)$ in the basis $\{\vec{u}_1, \vec{u}_2\}$ is (c_1, c_2)



Suggest we change coordinates into eigenbasis $\{\vec{u}_1, \vec{u}_2\}$

Let U be the matrix $\begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} = U\mathcal{B}$

$$AU = \begin{bmatrix} A\vec{u}_1 & A\vec{u}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1\vec{u}_1 & \lambda_2\vec{u}_2 \end{bmatrix}$$

$$= \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = U\Lambda$$

$$U^{-1}AU = \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

diagonalization of A.

New coordinates $\underline{y} = U^{-1}\underline{X}$

$$U\underline{y} = \underline{X}$$

Find our DE $\frac{d\underline{X}}{dt} = A\underline{X}$ in the $\underline{y}$-coord.

$$U\frac{d\underline{y}}{dt} = \frac{dU\underline{y}}{dt} = AU\underline{y}$$

$$\frac{d\underline{y}}{dt} = \left(U^{-1}AU\right)\underline{y} = \Lambda\,\underline{y} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\frac{dy_1}{dt} = \lambda_1 y_1 \qquad \frac{dy_2}{dt} = \lambda_2 y_2$$

$$y_1 = c_1 e^{\lambda_1 t} \qquad y_2 = c_2 e^{\lambda_2 t}$$

$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

decoupled

__Upshot__: Whenever the eigen val of A are $\lambda_1 \neq \lambda_2$ $\boxed{\text{nonzero}}$ real, then in some coord system the equation looks like

$$\frac{d\underline{y}}{dt} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \underline{y} \quad (*)$$

__and__ changing coord does not change stability, so it suffices to study equation $(*)$ in this __case__

A technique for computing 2x2 eigenvalues

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \begin{aligned} Tr(A) &= a + d = T \\ Det(A) &= ad - bc = D \end{aligned}$$

$$p(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

$$= (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + ad - bc$$

$$= \lambda^2 - T\lambda + D$$

Quad. formula:

$$\boxed{\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}}$$

CASE II eigenvalues of $A = \alpha \pm i\beta$ $\boxed{\beta \neq 0}$

Like in the previous case, there is a matrix U with

$$U^{-1}AU = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} = \Lambda$$

So we need only study

$$\frac{d\overline{X}}{dt} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \overline{X}$$

$$A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}, \text{ eigenvalues}$$

$$T = 2\alpha$$
$$D = \alpha^2 + \beta^2$$

$$\lambda = \frac{2\alpha \pm \sqrt{4\alpha^2 - 4(\alpha^2+\beta^2)}}{2}$$

$$= \frac{2\alpha \pm \sqrt{-4\beta^2}}{2} = \frac{2\alpha \pm i\beta \cdot 2}{2}$$

$$= \alpha \pm i\beta$$

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We study stability of $\vec{0}$

$$\frac{d\mathbf{X}}{dt} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \mathbf{X}$$

$$\boxed{\lambda = \alpha \pm i\beta}$$

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Use complex <u>notation</u>

$$z(t) = x(t) + i\,y(t)$$

$$\frac{dz}{dt} = \frac{dx}{dt} + i\,\frac{dy}{dt}$$

$$= (\alpha x - \beta y) + i\,(\beta x + \alpha y)$$

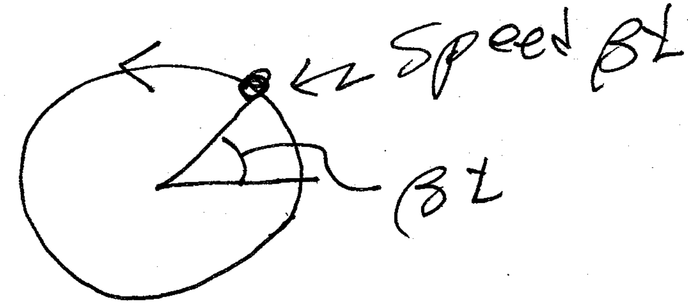
$$= (\alpha + i\beta)(x + iy) = \lambda z$$

DE becomes $\frac{dz}{dt} = \lambda z$ $\boxed{\lambda = \alpha + i\beta}$

<u>Solution</u> $z(t) = Ce^{\lambda t}$

$= Ce^{(\alpha + i\beta)t} = \left(Ce^{\alpha t}\right)\left(e^{i\beta t}\right)$

$= Ce^{\alpha t}\left[\underbrace{\cos\beta t}_{x} + i\underbrace{\sin\beta t}_{y}\right]$



amplitude $\alpha > 0$ $e^{\alpha t} \to \infty$

$\alpha < 0$ $e^{\alpha t} \to 0$

$\alpha = 0$, $e^{\alpha t} \equiv 1$

So soln is product of a decaying, constant or growing amplitude with uniform circular motion

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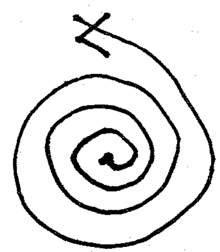
Stability and phase portraits

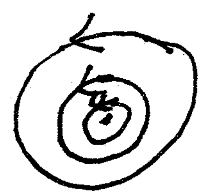
$\boxed{\alpha = \text{Real part}(\lambda)}$

$\alpha < 0$ - asymp stable spiral sink, attractor

$\alpha > 0$ - unstable, spiral, source, repeller.



$\alpha = 0$ - stable, <u>center</u>



Examples $\frac{d\bar{X}}{dt} = A\bar{X}$

(1) $A = \begin{bmatrix} -5 & -4 \\ 10 & 7 \end{bmatrix}$ $T = 2$ $D = -35 + 40 = 5$

$\lambda = \frac{2 \pm \sqrt{4 - 20}}{2}$

$= \frac{2 \pm 4i}{2} = 1 \pm 2i$

$\alpha = 1 > 0$ unstable spiral

(2) $A = \begin{bmatrix} -4 & -2 \\ 5 & 2 \end{bmatrix}$ $T = -2$ $D = -8 + 10 = 2$

$\lambda = \frac{-2 \pm \sqrt{4 - 8}}{2} = \frac{-2 \pm 2i}{2}$

$= -1 \pm i$

$\alpha = -1 < 0$

stable spiral.

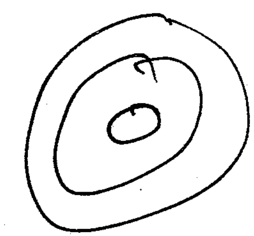
(3) $A = \begin{bmatrix} -6 & -4 \\ 10 & 6 \end{bmatrix}$ $T = 0$ $D = -36 + 40$

$= 4$

$\boxed{3}$

$$\lambda = \frac{0 \pm \sqrt{0^2 - \cancel{4}16}}{2}$$

$$= \frac{0 \pm 4i}{2} = 0 \pm 2i$$



$\cancel{\alpha}$ $d = 0$, so center, stable.