Schedule

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Fri 19 - TACK about exam, Review problems

Mon 22 - NO CLASS - WORK Review,

~~FR~~ Wed 24 - Review in class

1 hour exam - Tues 30, 3:00PM
<u>here</u>

$\boxed{\text{12.2 cont}}$ Stability of equilibrium.

for 2x2 Linear systems.

$$\frac{d\mathbf{X}}{dt} = A\mathbf{X}$$

λ_1, λ_2 the eigenvalues determine stability.

CASES

✓(I) $\lambda_1 \neq \lambda_2$ both real, & non zero.

✓(II) $\lambda = \alpha \pm \beta i$ $\beta \neq 0$

→ (III) $\lambda_1 = \lambda_2 \neq 0$

(IV) some $\lambda_i = 0$ (degenerate)

$T = Tr(A)$

$D = \det(A)$

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

(III). $\lambda_1 = \lambda_2 \neq 0$, Jordan decomposition

says that in this case there is a C

with $\underline{\underline{C^{-1}AC}} = $ (a) $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$ or (b) $\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}$

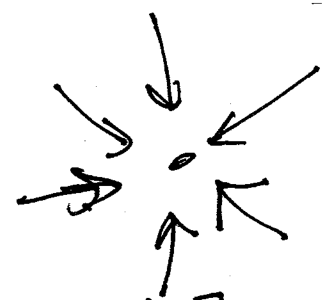
So in coordinates $y = C^{-1}x$ we just study

(a) $\frac{d\underset{\sim}{y}}{dt} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} \underset{\sim}{y}$ (b) $\frac{dy}{dt} = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \underset{\sim}{y}$

(a) $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\vec{x}(t) = c_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{\lambda_1 t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\lambda_2 = \lambda_1 < 0$ $\quad e^{\lambda_1 t} \to 0$ so asymp stability
sink



(b) $A = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$, char poly $p(\lambda) = (\lambda - \lambda_1)^2$

so $(A - \lambda_1 I)^2 = 0$ we compute

the matrix exp

$$e^{At} = e^{\lambda_1 t}\left(e^{A - \lambda t}\right) = e^{\lambda_1 t}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$$

$$= e^{\lambda_1 t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$\vec{x}(t) = c_1 \cancel{\left[e^{\lambda_1} \right.}$$

$$c_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{\lambda_1 t} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

$$\lambda_1 > 0, \quad e^{\lambda_1 t} \to \infty \quad (e^{\lambda_1 t}) \; t \to \infty$$

unstable, source

$$\boxed{\lambda_1 < 0} \quad e^{\lambda_1 t} \to 0, \quad t e^{\lambda_1 t} \to ?$$

$$\lim_{t \to \infty} t e^{\lambda_1 t} = \lim_{t \to \infty} \frac{t}{e^{-\lambda_1 t}} \overset{LHR}{=} \lim_{t \to \infty} \frac{1}{-\lambda_1 e^{-\lambda_1 t}}$$

$$= \lim_{t \to \infty} -\frac{1}{\lambda_1} e^{\lambda_1 t} = 0$$

asym. stab. sink.

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

double root only when $\underline{\underline{T^2 - 4D = 0}}$

$$\lambda = \frac{T}{2} \pm 0 = \frac{T}{2}$$

$\boxed{\text{examples}}$ $\frac{d\underline{X}}{dt} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \underline{X}$ $T = 4$, $D = 4$

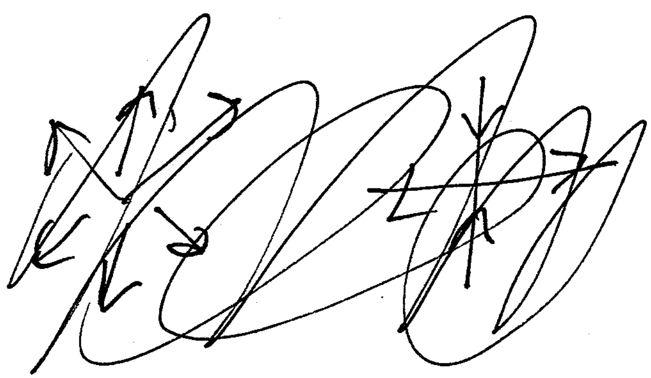
$$\lambda = \frac{4 \pm \sqrt{16 - 16}}{2} = 2$$ unstable source.

$\boxed{2}$

$$\frac{d\bar{X}}{dt} = \begin{bmatrix} -3 & -1 \\ 1 & -1 \end{bmatrix} \bar{X} \qquad T = -4 \quad D = 4$$

$$\lambda = \frac{-4 \pm \sqrt{16-16}}{2} = \underline{\underline{-2}}$$

~~as~~ s asym stable, sink



(IV) some $\lambda = 0$

happens when $\det(A) = 0$

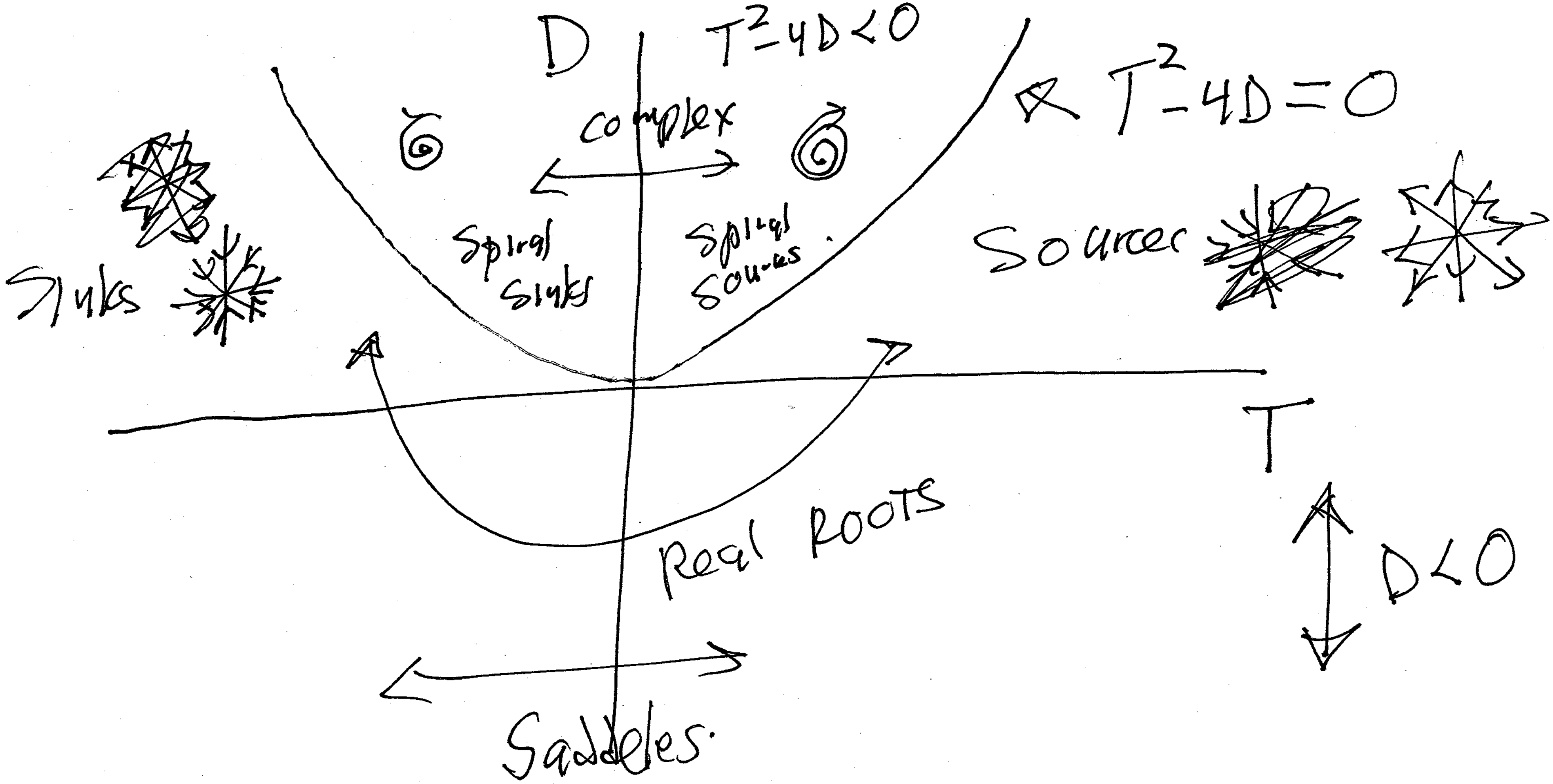
$$\lambda = \frac{T \pm \sqrt{T^2 - 4 \cdot 0}}{2} = \frac{T \pm T}{2} = T, 0$$

This implies a line of fixed point since there is a eigenvector $\vec{u}$ with $A\vec{u} = 0\vec{u}$ so

$\frac{dX}{dt} = AX$, $\frac{dX}{dt}$ vanishes on the line $\vec{u}$

Stability ~~or~~ Birfurcation diagram

$$\lambda = \left(T \pm \sqrt{T^2 - 4D}\right)/2$$



12.2 - More general DE - stability of equilibrium

$$\frac{d\bar{X}}{dt} = A\bar{X} + \vec{q}$$

$\vec{q}$ constant vector

equilibrium $\vec{0} = \frac{d\bar{X}}{dt} = A\bar{X} + \vec{q}$

$$A\bar{X} = -\vec{q}, \quad \bar{X} = -A^{-1}\vec{q}$$

eg// Find the equilibrium and its stasity for

$$\frac{d\underline{X}}{dt} = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix} \underline{X} + \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

Soln: ~~equib~~ equilibrium

$$\vec{0} = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$\begin{aligned} 2x + y &= -3 \\ -3x - 2y &= 4 \end{aligned} \Rightarrow \underline{X}_0 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$ the equilibrium

Move the origin to $\underline{X}_0$ via change of coord.

$$u = x + 2 \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v = y - 1$$

$$\downarrow$$

$$\boxed{u - 2 = x \quad v + 1 = y}$$

Now rewrite the DE in terms of (u, v).

$$\frac{dx}{dt} = 2x + y + 3 \qquad \frac{dy}{dt} = -3x - 2y - 4$$

$$\frac{du}{dt} = \frac{dx}{dt} = 2(u-2) + (v+1) + 3 = 2u + v$$

$$\frac{dv}{dt} = \frac{dy}{dt} = -3(u-2) - 2(v+1) - 4 = -3u - 2v$$

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \quad T=0 \quad D=-1$$

$$\lambda = \frac{0 \pm \sqrt{0+4}}{2} = \pm 1$$

Saddle, unstable for $\begin{pmatrix} u \\ v \end{pmatrix}$ and

Thus for $\begin{pmatrix} x \\ y \end{pmatrix}$