$\boxed{12.2}$ Review stability of $\vec{0}$ for $\frac{dX}{dt} = AX$ 1

two-dimensional

A has eigenvalues λ_1, λ_2 $\boxed{\text{both non-zero}}$

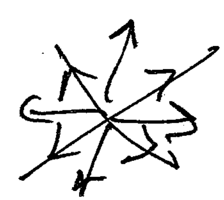
(I) $\lambda_1 \neq \lambda_2$ real

(a) $\lambda_1 < \lambda_2 < 0$ sink, asym stable

(b) $\lambda_1 < 0 < \lambda_2$ saddle, unstable

(c) $0 < \lambda_1 < \lambda_2$, source, unstable





(a) (b) (c)

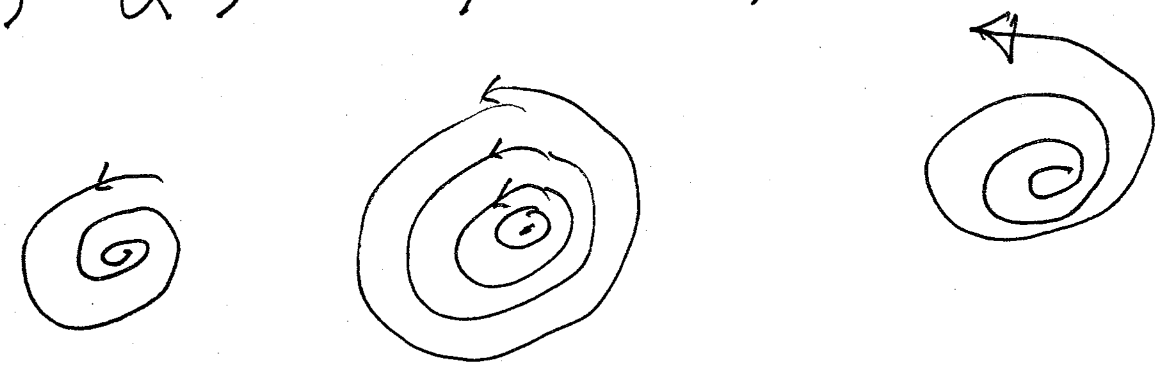
2

II) $\lambda = \alpha + i\beta \quad \beta \neq 0$

(a) $\alpha < 0$, sink, asympstable, spiral

(b) $\alpha = 0$, center, stable [pure imag]

(c) $\alpha > 0$, source, unstable, spiral



(a) (b) (c)

III $\lambda_1 = \lambda_2$ real

(a) $\lambda_1 = \lambda_2 < 0$ sink, asymp stable

(b) $\lambda_1 = \lambda_2 > 0$ Source, unstable.

$\boxed{\text{General Rule}}$

$Re(\lambda) < 0$ all $\lambda \Rightarrow$ asymp stable

real part [all eigenvalues are in the left half plane]

Any λ with $Re(\lambda) > 0 \Rightarrow$ unstable.

$Re(\lambda) = 0$, Special case.

Same general Rule works in higher dimensions $\frac{d\bar{X}}{dt} = A\bar{X}$

but $\bar{X}$ is n-dimensional

This WAS all Linear stability

What about more general DE

or <u>nonlinear stability</u>.

The main tool is the linear stability of the derivative at the equilibrium

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Set up

$$\left.\begin{aligned} \frac{dx}{dt} &= f(x,y) \\ \frac{dy}{dt} &= g(x,y) \end{aligned}\right\} \text{DE}$$

$$\underline{X} = (x,y) \quad \vec{F}(x,y) = (f(x,y), g(x,y))$$

$$\frac{d\underline{X}}{dt} = \vec{F}(\underline{X})$$

equilibrium is where $\vec{F}(\underline{X}_0)$

$$\frac{d\underline{X}}{dt} = 0 \quad \text{at } \underline{X}_0 .$$

Taylor's Theorem in 2D

$$\vec{F} = (f, g)$$

$$D\vec{F} = \nabla \vec{F} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

Taylor expansion about $\underline{X}_0$.

$$\vec{F}(\underline{X}) = \vec{F}(\underline{X}_0) + \nabla F(\underline{X}_0)(\underline{X}-\underline{X}_0) + \text{h.o.t}$$

h.o.t = higher order terms.

at an equilibrium $\vec{F}(\underline{X}_0)$

So $\vec{F}(\underline{X}) = \underline{\underline{\nabla F(\underline{X}_0)}}(\underline{X}-\underline{X}_0) + \ldots$

In $\underline{y} = \underline{X} - \underline{X}_0$

$$\vec{F} \approx \nabla F(\underline{X}_0)\,\underline{y}$$ to first order.

So near $\underline{X}_0$, DE looks like

$$\boxed{\frac{d\underline{y}}{dt} = \nabla F(\underline{X}_0)\,\underline{y}}$$

Linearization about $\underline{X}_0$.

Stability of ~~this~~ $\underline{y} = 0$ in this eq. tells us about the stability of $\underline{X}_0$.

Example $\dfrac{dx}{dt} = 7x - x^2 - 2xy = f(x,y)$

$$\frac{dy}{dt} = 5y - y^2 - xy = g(x,y)$$

has 4 equilibrium computed later, For now just consider $\underline{X}_0 = (3,2)$. $\vec{F}(\underline{X}_0) = 0$

$$\nabla \vec{F} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 7-2x-2y & -2x \\ -y & 5-2y-x \end{bmatrix}$$

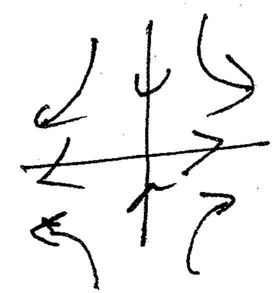
$$\nabla \vec{F}(3,2) = \begin{bmatrix} 7-6-4 & -6 \\ -2 & 5-4-3 \end{bmatrix} = \begin{bmatrix} -3 & -6 \\ -2 & -2 \end{bmatrix}$$

eigenvalues $\begin{bmatrix} -3 & -6 \\ -2 & -2 \end{bmatrix}$ $T = -5$

$D = 6 - 12 = -6$

$$\lambda = \frac{-5 \pm \sqrt{25+24}}{2}$$

$$= \frac{-5 \pm 7}{2} = 1, -6.$$



Linear ~~of~~ saddl, unstable.

By the nonlinear stability Th'm,

(3,2) is also an unstable, saddle.

The Non Linear Stability Theorem.

Except for the case $\lambda = 0 \pm i\beta$ and $\lambda = 0$.
the nonlinear stability and type is
the same as its linearization
from $\nabla\vec{F}(\mathbf{X}_0)$ at equilibrium $\mathbf{X}_0$.

eg// evaluate the stability of $(0,0)$ as an equilibrium of

$$\frac{dx}{dt} = -x + 4y + y^2 = f$$

$$\frac{dy}{dt} = -x - y - xy = g$$

$$\nabla F = \begin{bmatrix} -1 & 4+2y \\ -1-y & -1-x \end{bmatrix}$$

$$\nabla F(0,0) = \begin{bmatrix} -1 & 4 \\ -1 & -1 \end{bmatrix}$$

$T = -2$

$D = 5$

$$\lambda = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= -1 \pm 2i$$

<u>Linearization</u> ~~stable a~~ asympstable sink (spiral)

So this is also the nonlinear stability

~~Proc~~ Procedure

1. Find the equilibrium by solving $\vec{F}(\vec{X}) = 0$
2. Compute $\nabla \vec{F}$
3. For each equilibrium X_0, compute $\nabla F(X_0)$ and its eigenvalues and stability.
4. Us the Nonlinear Stability Th'm for the full answer.

Ex: Find the equilibrium and evaluate their stability for

$$\frac{dx}{dt} = 7x - x^2 - 2yx$$

$$\frac{dy}{dt} = 5y - y^2 - xy$$

$$x(7 - x - 2y) = 7x - x^2 - 2xy = 0$$

$$y(5 - y - x) = 5y - y^2 - xy = 0$$

4 cases

(1) $x = y = 0$

(2) $x = 0$, $5 - y - x = 0$, $5 - y = 0$ $y = 5$

(3) $y = 0$, $7 - x - 2y = 0$, $7 - x = 0$, $x = 7$

(4) $7 - x - 2y = 0$ $5 - y - x = 0$

$x + 2y = 7$ $x = 3$ $y = 2$.

$x + y = 5$

Four equilibrn $\boxed{(0,0), (0,5), (7,0), (3,2)}$