Your Name Printed Clearly! Key

Justify all answers! Show all work for partial credit! You will get no credit for just the answer. Note that different problems have different values. No calculators, no notes.

EXAM 1 • BOYLAND • MAP4305 • SPRING 2019

1.(20 *points*)Find the first **four** nonzero terms in the power series expansion about $x_0 = 0$ for the general solution to the given differential equation. Your answer must include the recursion relation for the coefficients.

$$y'' - 2xy' + y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

plug into equation

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - 2x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

or $$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Big\Downarrow \boxed{k=n-2}$$

$$\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) x^k - \sum_{k=1}^{\infty} 2k a_k x^k + \sum_{k=0}^{\infty} a_k x^k = 0$$

$k=0$ yields $2a_2 + a_0 = 0$ so $a_2 = -a_0/2$

$k>0$ yields $(k+2)(k+1)a_{k+2} - 2k a_k + a_k = 0$

so recursion relation $\boxed{a_{k+2} = \frac{(2k-1)a_k}{(k+1)(k+2)}}$

$k=1$ yields $a_3 = \frac{1}{6} a_1$ so

$$y(x) = a_0\left(1 - \tfrac{1}{2}x^2 + \cdots\right) + a_1\left(x + \tfrac{1}{6}x^3 + \cdots\right)$$

Total points this page out of 20 ________

Key

2.(10 *points*) Determine the first **four** nonzero terms in the Taylor polynomial approximation to the given initial value problem.

$$y'' + y' + \sin(x + y - 1) = 0; \quad y(0) = \pi, \ y'(0) = 1$$

$y(0) = \pi$

$y'(0) = 1$

$y''(0) = -y'(0) - \sin(0+\pi-1) = -1 - \sin(\pi - 1)$

$y'''(x) = -y''(x) - \cos(x+y-1)(1+y')$

$y'''(0) = 1 + \sin(\pi-1) - \cos(\pi-1)\cdot 2$

$$y(x) = \pi + x - (1+\sin(\pi-1))\frac{x^2}{2} + \left[1 + \sin(\pi-1) - 2\cos(\pi-1)\right]\frac{x^3}{6} + \cdots$$

3.(5 *points*) Write the general solution using Bessel functions

$$4x^2y'' + 4xy' + (4x^2 - 9)y = 0$$

$x^2 y'' + x y' + \left(x^2 - \frac{9}{4}\right)y = 0$

$\nu^2 = 9/4 \quad \nu = \pm 3/2$

$y(x) = c_1 J_{3/2}(x) + c_2 J_{-3/2}(x)$

Total points this page out of 15 ________

key

4.(15 *points*) For this differential equation

$$(x+1)x^2y'' - xy' + (x+1)y = 0 \Rightarrow y'' - \frac{1}{x(x+1)}y' + \frac{1}{x^2}y = 0$$

(a) Find and classify all singular points as regular or irregular.

$p = -\frac{1}{x(x+1)}$ $q = 1/x^2$ so sing pts at $x_0 = 0$, $x_0 = -1$

$\boxed{x_0 = 0}$ $xp = -1/(x+1)$ at $x=0$, $p_0 = -1$, $x^2 q = 1$ so $q_0 = 1$ $\boxed{\text{Regular}}$

$(x_0 = -1)$ $(x+1)p = -1/x$ at $x = -1$, $p_0 = 1$, $(x+1)^2 q = \frac{(x+1)^2}{x^2}$ at $x=-1$, $q_0 = 0$ $\boxed{\text{Regular}}$

(b)Find the indicial equation and exponents at the singularity $x_0 = 0$.

For $x_0 = 0$ from (a) $p_0 = -1$, $q_0 = 1$ so

$r(r-1) - r + 1 = 0$, $r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$ $r = 1$ with multiplicity 2.

(c) Write down the **form** of the series for one solution at the singularity $x_0 = 0$.

$$y_1(x) = x^1 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1}$$

(d)What is the minimum radius of convergence of the series in (c) and why?

distance from $x_0 = 0$ to nearest other singularity -1 is 1, so min. rad. of conv = 1

(e)Write down the **form** of the series for the second linearly independent solution at the singularity $x_0 = 0$

Since double root

$$y_2(x) = y_1(x)\ln x + x\sum_{n=0}^{\infty} b_n x^n$$

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5.(20 *points*) Find the power series expansion about $x_0 = 0$ for the general solution to the given differential equation. Your answer should be in the form of a summation including a general formula for the coefficients. Your answer must include the recursion relation for the coefficients.

$$y'' - 2y = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - \sum_{n=0}^{\infty} 2 a_n x^n = 0$$

$$\boxed{k = n-2}$$

$$\sum_{k=0}^{\infty} a_{k+2} (k+2)(k+1) x^k - \sum_{k=0}^{\infty} 2 a_k x^k = 0$$

$$a_{k+2} = \frac{2a_k}{(k+1)(k+2)} \quad \text{Recursion}$$

a_0, a_1 are constants

$k=0 \quad a_2 = \frac{2a_0}{1\cdot 2}$

$k=1 \quad a_3 = \frac{2a_1}{2\cdot 3}$

$k=2 \quad a_4 = \frac{2a_2}{3\cdot 4} = \frac{2^2 a_0}{3\cdot 4\cdot 1\cdot 2}$

$k=3 \quad a_5 = \frac{2a_3}{4\cdot 5} = \frac{2^2 a_1}{2\cdot 3\cdot 4\cdot 5}$

$$a_{2m} = \frac{2^m}{(2m)!} a_0 \qquad a_{2m+1} = \frac{2^m}{(2m+1)!} a_1$$

$$y(x) = a_0 \sum_{m=0}^{\infty} \frac{2^m x^{2m}}{(2m)!} + a_1 \sum_{m=0}^{\infty} \frac{2^m x^{2m+1}}{(2m+1)!}$$

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