Your Name Printed Clearly! Key

Justify all answers! Show all work for partial credit! You will get no credit for just the answer. Note that different problems have different values. No calculators, no notes. The formula sheet is page 6.

EXAM 2 • BOYLAND • MAP4305 • SPRING 2019

1.(10 *points*) Find a fundamental matrix for:

$$x' = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix} x$$

$$p(\lambda) = (4-\lambda)(2-\lambda) + 1 = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$$

Thus $(A-3I)^2 = 0$ and we can compute a fund. matrix using the matrix exponential

$$e^{At} = e^{3t}\left[I + t(A-3I)\right]$$

$$= e^{3t}\left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\right]$$

$$= e^{3t}\begin{bmatrix} 1+t & t \\ -t & 1-t \end{bmatrix}$$

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2.(15 *points*)

(a) Find the general solution to

$$\mathbf{x}' = \begin{pmatrix} -8 & -10 \\ 5 & 7 \end{pmatrix} \mathbf{x}$$

(b) Find the general solution to

$$\mathbf{x}' = \begin{pmatrix} -8 & -10 \\ 5 & 7 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -6 \\ 3 \end{pmatrix}$$

(a) $p(\lambda) = (-8-\lambda)(7-\lambda) + 50 = \lambda^2 + \lambda - 56 + 50$

$= \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2)$

$\lambda = 2$, $0 = (A - 2I)\vec{u} = \begin{pmatrix} -10 & -10 \\ 5 & 5 \end{pmatrix} \vec{u}$ so $u_1 + u_2 = 0$ so $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda = -3$, $0 = (A + 3I)\vec{u} = \begin{pmatrix} -5 & -10 \\ 5 & 10 \end{pmatrix} \vec{u}$ so $u_1 + 2u_2 = 0$

$\vec{u}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\vec{x}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

(b) The particular soln $\vec{x}_p = \vec{a}$, $\vec{x}_p' = 0 =$

$\begin{pmatrix} -8 & -10 \\ 5 & 7 \end{pmatrix} \vec{a} + \begin{pmatrix} -6 \\ 3 \end{pmatrix}$ so $\begin{pmatrix} -8 & -10 \\ 5 & 7 \end{pmatrix} \vec{a} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$

$\vec{a} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ so $\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$

$= c_1 e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

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3.(10 *points*) Solve this equation on $0 \le x \le \pi$:

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$$
$$u(0,t) = u(\pi,t) = 0$$
$$u(x,0) = 2\sin 3x$$
$$\frac{\partial u}{\partial t}(x,0) = -3\sin 2x + 4\sin 5x$$

WAVE equation $\alpha = 2$, $L = \pi$.

$$2\sin 3x = \sum_{n=1}^{\infty} a_n \sin nx \quad \text{so } a_3 = 2.$$

$$-3\sin 2x + 4\sin 5x = \sum_{n=1}^{\infty} b_n (n\alpha)\sin(nx)$$

$$\text{so} \quad -3 = b_2(2\cdot 2), \quad b_2 = -3/4$$
$$4 = b_5(5\cdot 2), \quad b_5 = 2/5$$

$$\text{So} \quad u(x,t) = 2\cos(3\cdot 2)t\,\sin(3x)$$
$$-3/4\sin(2\cdot 2)t\,\sin 2x$$
$$+2/5\sin(5\cdot 2)t\,\sin 5x$$
$$= 2\cos 6t\,\sin 3x - 3/4\sin 4t\,\sin 2x$$
$$+2/5\sin(10t)\,\sin 5x$$

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4.(15 *points*)

(a) Find the general solution

$$\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 2 & 3 \end{pmatrix} \mathbf{x}$$

(b) Find the solution that satisfies the given initial condition

$$\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 2 & 3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$

(a) $p(\lambda) = (-1-\lambda)(3-\lambda) + 8 = \lambda^2 - 2\lambda + 5$

$\lambda = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$

$0 = (A - \lambda I)\vec{u} = \begin{pmatrix} -1-(1+2i) & -4 \\ 2 & 3-(1+2i) \end{pmatrix}\vec{u} = \begin{pmatrix} -2-2i & -4 \\ 2 & 2-2i \end{pmatrix}\vec{u}$

$(-2-2i)u_1 - 4u_2 = 0$, Let $u_1 = 2$ then

$-4-4i = 4u_2$ or $u_2 = -1-i$

So $\alpha = 1, \beta = 2, \vec{a} + i\vec{b} = \begin{bmatrix} 2 \\ -1-i \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + i\begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$x_1(t) = e^t \cos 2t \begin{bmatrix} 2 \\ -1 \end{bmatrix} - e^t \sin 2t \begin{bmatrix} 0 \\ -1 \end{bmatrix} = e^t \begin{bmatrix} 2\cos 2t \\ -\cos 2t + \sin 2t \end{bmatrix}$

$x_2(t) = e^t \sin 2t \begin{bmatrix} 2 \\ -1 \end{bmatrix} + e^t \cos 2t \begin{bmatrix} 0 \\ -1 \end{bmatrix} = e^t \begin{bmatrix} 2\sin 2t \\ -\sin 2t - \cos 2t \end{bmatrix}$

(b) $\begin{pmatrix} -2 \\ -2 \end{pmatrix} = \vec{x}(0) = c_1 x_1(0) + c_2 x_2(0) = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$= c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

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So $2c_1 = -2$, $c_1 = -1$, $c_2 = +3$, $\vec{x}(t) = -e^t \begin{bmatrix} 2\cos 2t \\ -\cos 2t + \sin 2t \end{bmatrix} + 3e^t \begin{bmatrix} 2\sin 2t \\ -\sin 2t - \cos 2t \end{bmatrix}$

Another Solution to 4

Starting with $0 = \begin{pmatrix} -2-2i & -4 \\ 2 & 2-2i \end{pmatrix} \vec{u}$

The 2^{nd} equation is $2u_1 + (2-2i)u_2 = 0$

or $u_1 = (-1+i)u_2$ $\quad u_2 = 1$ $\quad u_2 = -1+i$

$\vec{u} = \begin{bmatrix} -1+i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

so $x_1(t) = e^t \cos t \begin{bmatrix} -1 \\ 1 \end{bmatrix} - e^t \sin t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$x_2(t) = e^t \sin 2t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + e^t \cos t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{x}(t) = c_1 x_1(t) + c_2 x_2(t)$

(b) $\begin{pmatrix} -2 \\ -2 \end{pmatrix} = c_1 x_1(0) + c_2 x_2(0) = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

so $c_1 = -2$ $\quad c_2 = -4$

5.(10 *points*) Compute the Fourier **sine** series for $f(x) = 1 + x$ on $0 \leq x \leq \pi$. Note that your coefficients must be numbers, so expressions like $\cos n\pi/2$ are *not* allowed and must be evaluated to a numerical value.

We do the two terms separately.

For 1, $b_n = \frac{2}{\pi}\int_0^\pi 1 \sin nx\, dx = \frac{2}{\pi} \frac{-\cos nx}{n}\Big|_0^\pi$

$= \frac{2}{\pi n}(-\cos n\pi + 1) = \boxed{2\left[\frac{1+(-1)^{n+1}}{n\pi}\right]}$

for x, $b_n = \frac{2}{\pi}\int_0^\pi x \sin nx\, dx = \frac{2}{\pi}\int_0^{n\pi} \frac{u}{n} \sin u \frac{du}{n}$

$u = nx$
$du = n\,dx$

$= \frac{2}{n^2\pi}\left[\sin u - u\cos u\Big|_0^{n\pi}\right] = \frac{2}{\pi n^2}\Big[\sin n\pi - n\pi\cos n\pi$

$- (\sin 0 \cdot 0\cos 0)\Big] = \frac{2}{\pi n^2} n\pi(-1)^{n+1} = \boxed{\frac{2}{n}(-1)^{n+1}}$

So for $1+x = f(x)$, $b_n = \frac{2(1+(-1)^{n+1})}{n\pi} + \frac{2}{n}(-1)^{n+1}$

$= 2\left(\frac{1+(-1)^{n+1} + \pi(-1)^{n+1}}{n\pi}\right) = \frac{1+(-1)^{n+1}(1+\pi)}{n\pi}$ so the sine series is

$$\boxed{2\sum_{n=1}^{\infty} \left(\frac{1+(-1)^{n+1}(1+\pi)}{n\pi}\right) \sin(nx).}$$

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