MAA4402 • FALL 2017 • EXAM # 2 • PROF. BOYLAND

Justify all answers! Show all work! You will get no credit for just the answer. Problems have different point value

1.(15 *points*) Compute

$$\oint_C \frac{\cos(z)}{z^2(z-\pi)}\, dz$$

for these contours C all with $0 \le t \le 2\pi$

$z=0$ is order 2 pole

$\phi(z) := \cos z/(z-\pi)$

$\phi'(z) = \dfrac{(z-\pi)(-\sin z) - \cos z}{(z-\pi)^2}$

$\operatorname{Res}_{z=0} \dfrac{\phi(z)}{z^2} = -\dfrac{1}{\pi^2}$

$z=\pi$ is simple pole

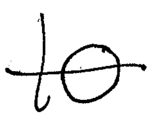
$\phi(z) := \cos z/z^2$

$\operatorname{Res}_{z=\pi} \dfrac{\phi(z)}{z-\pi} = \dfrac{\cos\pi}{\pi^2} = -\dfrac{1}{\pi^2}$

(a) C: $z(t) = e^{it}$ just zero inside contour

$\oint = (2\pi i)\left(-\frac{1}{\pi^2}\right) = \frac{-2i}{\pi}$

(b) C: $z(t) = e^{it} + \pi$ just π inside the contour



$\oint = (2\pi i)\left(-\frac{1}{\pi^2}\right) = \frac{-2i}{\pi}$

(c) C: $z(t) = e^{it} - \pi$ no sing inside contour so

$\oint = 0$

(d) C: $z(t) = 4e^{it}$ both sing inside contour so

$\oint = 2\pi i\left(-\frac{1}{\pi^2} + -\frac{1}{\pi^2}\right) = \frac{-4i}{\pi}$

$$f(z) = -\frac{1}{z-1} + \frac{1}{z-2}$$

2.(15 *points*) Let $f(z) = \frac{1}{(z-1)(z-2)}$. Find the Laurent series expanded about zero in each of the following regions.

(a) $0 < |z| < 1$

$$f(z) = \frac{1}{1-z} - \frac{1}{2-z} = \frac{1}{1-z} - \frac{1}{2}\left(\frac{1}{1-\frac{z}{2}}\right)$$

$$= \sum_{n=0}^{\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty}\left(1 - \frac{1}{2^{n+1}}\right)z^n$$

$|z| < 1$ and $\left|\frac{z}{2}\right| < 1$ so $|z| < 2$

(b) $1 < |z| < 2$

$$f(z) = -\frac{1}{z}\left[\frac{1}{1-\frac{1}{z}}\right] - \frac{1}{2}\left(\frac{1}{1-\frac{z}{2}}\right)$$

$$= -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$\left|\frac{1}{z}\right| < 1$ or $1 < |z|$

$$= -\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$$

$\left|\frac{z}{2}\right| < 1$ or $|z| < 2$

(c) $2 < |z|$.

$$f(z) = -\frac{1}{z}\left[\frac{1}{1-\frac{1}{z}}\right] + \frac{1}{z}\left[\frac{1}{1-\frac{2}{z}}\right]$$

$$= -\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}$$

$\left|\frac{1}{z}\right| < 1$ or $|z| > 1$

$\left|\frac{2}{z}\right| < 1$ or $|z| > 2$

$$= \sum_{n=1}^{\infty}\frac{-1 + 2^{n-1}}{z^n}$$

3.(20 *points*)

(a) Find the first 3 nonzero terms of the Maclaurin seres of

$$f(z) = z\sin(z^2)$$

$\sin z = z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots$

$\sin z^2 = z^2 - \frac{z^6}{6} + \frac{z^{10}}{120} - \cdots$

$z\sin(z^2) = z^3 - \frac{z^7}{6} + \frac{z^{11}}{120} + \cdots$

or by differentiating $-\frac{1}{2}\cos(z^2)$

(b) Compute

$$\oint_{|z|=4} \frac{e^z}{\sin(z)}\, dz$$

$\sin(z) = 0$ when $z_n = n\pi$ $\quad p(z) = e^z \quad q(z) = \sin z$

$q'(z) = \cos z$

$\operatorname*{Res}_{z=z_n} \frac{e^z}{\sin z} = \frac{e^{n\pi}}{\cos n\pi}$ (simple poles) $\quad \cos(z_n) \neq 0$

z_0, z_1, z_{-1} are inside the contour so

$\oint_{|z|=4} \frac{e^z}{\sin z}\, dz = 2\pi i \left[\frac{e^{-\pi}}{-1} + \frac{1}{1} + \frac{e^{\pi}}{-1}\right]$

$= 2\pi i\left[1 - e^{-\pi} - e^{\pi}\right]$

(c) Compute

$$\oint_{|z|=1} z\cos(1/z)\, dz$$

Computing Laurent series

$\cos z = 1 - \frac{z^2}{2} + \frac{z^4}{12} + \cdots$

$\cos \frac{1}{z} = 1 - \frac{1}{2z^2} + \frac{1}{12z^4} + \cdots$

$z\cos\frac{1}{z} = z - \frac{1}{2}\frac{1}{z} + \frac{1}{12z^3} + \cdots$

So $\operatorname*{Res}_{z=0} z\cos\frac{1}{z} = -\frac{1}{2}$

(essential singularity)

so $\oint_{|z|=1} z\cos(\frac{1}{z})\, dz = (2\pi i)(-\frac{1}{2})$

$= -\pi i$

4.(15 *points*) Compute this integral using residues. Be sure to justify all your steps carefully.

$$\int_0^\infty \frac{dx}{(2x)^2+1}$$

(1) $\frac{1}{(2x)^2+1}$ is even, so $\int_0^\infty \frac{dx}{(2x)^2+1} = \frac{1}{2}\, PV \int_{-\infty}^{\infty} \frac{dx}{(2x)^2+1}$

(2) Let Γ_R be as shown

$R > \frac{1}{2}$

and

$$\oint_{\Gamma_R} \frac{1}{(2z)^2+1}\,dz = \int_{-R}^{R} \frac{dx}{(2x)^2+1} + \int_{C_R} \frac{dz}{(2z)^2+1}$$

(3) $(2z)^2+1=0$ or $2z = \pm i$ $\quad z = \pm\frac{i}{2}$, just $\frac{i}{2}$ is sing inside the contour

$p = 1$
$q(z) = (2z)^2+1 \quad q'(z) = 4z \quad 4(\frac{i}{2}) \neq 0$ so simple pole

$$\operatorname*{Res}_{z=\frac{i}{2}} \frac{1}{(2z)^2+1} = \frac{1}{8(\frac{i}{2})} = \frac{1}{4i}$$

So $$\oint_{\Gamma_R} \frac{1}{(2z)^2+1}\,dz = (2\pi i)\frac{1}{4i} = \frac{\pi}{2}$$

(4) On C_R we have $\left|\int_{C_R} \frac{dz}{(2z)^2+1}\right| \leq (\pi R)\frac{1}{|2R^2-1|}$

by the integral and reverse triangle inequality so

$$0 \leq \lim_{R\to\infty}\left|\int_{C_R} \frac{dz}{(2z)^2+1}\right| \leq \lim_{R\to\infty} \frac{\pi R}{2R^2-1} = 0 \text{ so } \lim_{R\to\infty}\int_{C_R} \frac{dz}{(2z)^2+1} = 0$$

(5) Taking the limit in (2)

$\pi = PV\int_{-\infty}^{\infty} \frac{dx}{(2x)^2+1}$ so by (1), $\int_0^\infty \frac{dx}{(2x)^2+1} = \frac{\pi}{4}$.

Exam #2 - Complex
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