MAA4402 • FALL 2017 • EXAM # 1 • PROF. BOYLAND

Justify all answers! Show all work! You will get no credit for just the answer. All problems have the same point value

1. Show that

$$f(x+iy) = x + \cos(x)\cosh(y) + i(y - \sin(x)\sinh(y))$$

is an entire function.

$u = x + \cos x \cosh y$

$v = y - \sin x \sinh y$

$u_x = 1 - \sin x \cosh y \qquad v_y = 1 - \sin x \cosh y$ ✓

$u_y = \cos x \sinh y \qquad v_x = -\cos x \sinh y$ ✓

so CR equations satisfied everywhere

so entire since all partials exist and are continuous

2. Find all values of $(-1+i)^{3i}$. Which is the principle value?

$(-1+i)^{3i} = \exp[\log(-1+i)\cdot 3i]$ $\qquad n = 0, \pm 1, \pm 2$

$= \exp[\ln\sqrt{2} + i(\frac{3\pi}{4} + 2n\pi)\cdot 3i]$

$= \exp(-3(\frac{3\pi}{4} + 2n\pi) + 3i\ln\sqrt{2})$

$= \exp(-\frac{9\pi}{4} - 6n\pi)\exp(\frac{3i}{2}\ln 2)$

PV: $n=0 \qquad e^{-9\pi/4} e^{i3\ln\sqrt{2}} = e^{-9\pi/4}(\cos 3\ln\sqrt{2} + i\sin 3\ln\sqrt{2})$

3. Show using the definitions in terms of exponentials,

$$1 = \cosh^2(z) - \sinh^2(z).$$

$$\cosh^2 z - \sinh^2 z = \left(\frac{e^{z}+e^{-z}}{2}\right)^2 - \left(\frac{e^{z}-e^{-z}}{2}\right)^2$$

$$= \frac{1}{4}\left[e^{2z}+2+e^{-2z} - (e^{2z}-2+e^{-2z})\right] = \frac{4}{4} = 1$$

4. Let C be path $z(t) = 2e^{it}$ for $-\pi \le t \le \pi$. Compute

$$\oint_C z\bar{z}^2\, dz.$$

$$f(z(t)) = 2e^{it} \cdot 4e^{-2it} = 8e^{-it}$$

$$z'(t) = 2ie^{it}$$

$$\int_C f\, dz = \int_{-\pi}^{\pi} 8e^{-it} \cdot 2i\, e^{it}$$

$$= \int_{-\pi}^{\pi} 16i\, dt = 16i \cdot 2\pi = 32\pi i$$

5. Show this is true or else give an example that shows it isn't true: for all z_1, z_2,

$$\mathrm{Log}(z_1 z_2) = \mathrm{Log}(z_1) + \mathrm{Log}(z_2)$$

$z_1 = e^{\frac{3\pi}{4} i} \quad z_2 = e^{3\pi/4 i}$

$\mathrm{Log}(z_1) = \mathrm{Log}(z_2) = \frac{3\pi}{4} i$

So $\mathrm{Log}(z_1) + \mathrm{Log}(z_2) = \frac{6\pi}{4} i = \frac{3\pi}{2} i$

whereas $\mathrm{Log}(z_1 z_2) = \mathrm{Log}(e^{\frac{6\pi}{4} i}) = -\frac{\pi}{2} i$

So not true

6. Using the antiderivative,

$$\int_{\gamma} z e^{iz^2}\, dz,$$

where γ is a path from $-\sqrt{\frac{\pi}{2}}$ to $\sqrt{\frac{\pi}{2}}$. You must put the answer in rectangular form.

$f(z) = z e^{iz^2} \quad F(z) = \frac{e^{iz^2}}{2i}$

$\int_{\gamma} z e^{iz^2} dz = \frac{e^{iz^2}}{2i} \Big|_{-\sqrt{\pi/2}}^{\sqrt{\pi/2}} = \frac{e^{i\pi/2} - e^{i\pi/2}}{2i} = 0$

7. Find all values of $(-8 - i8\sqrt{3})^{1/4}$. You must put the answer in rectangular form.

$$-8-8\sqrt{3} = 16\, e^{(-\frac{2}{3}\pi i + 2n\pi i)}$$

$$(-8-8\sqrt{3})^{1/4} = 16^{\frac{1}{4}} \exp\left(\left(-\frac{2}{3}\pi i + 2n\pi i\right)\tfrac{1}{4}\right)$$

$$= 2\exp\left(-\frac{\pi i}{6} + \frac{n\pi}{2}\right)$$

$n=0$: $2\, e^{-\pi i/6} = 2\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = \sqrt{3} - i$

$n=1$: $2\, e^{-\frac{\pi i}{6} + \frac{\pi}{2} i} = 2e^{\frac{\pi}{3} i} = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) = 1 + \sqrt{3} i$

$n=2$: $2\, e^{-\frac{\pi i}{6} + \pi} = 2e^{5\pi/6\, i} = -\sqrt{3} + i$

$n=3$: $2\, e^{-\frac{\pi i}{6} + \frac{3\pi}{2}} = 2e^{4/3 \pi i} = -1 - \sqrt{3} i$

8. Show that

$$|\mathrm{Re}(2z^3 - i3z + 4)| \leq 9$$

when $|z| \leq 1$.

Since for any w $|\mathrm{Re}(w)| \leq |w|$ we get

$$|\mathrm{Re}(2z^3 - i3z + 4)| \leq |2z^3 - 3iz + 4|$$

$$\leq |2z^3| + |3iz| + |4|$$

Triangle inequality

$$= 2|z|^3 + |3||i||z| + |4|$$

$$= 2|z^3| + 3|z| + 4$$

$$\leq 2 + 3 + 4 = 9$$

since $|z| \leq 1$