Your Name Printed Clearly! Key

Justify all answers! Show all work for partial credit! You will get no credit for just the answer. Different problems have different values. No calculators, no notes

EXAM 1 • BOYLAND • LADS • SPRING 2020

1.(15 *points*) Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

(a) Compute the eigenvalues and eigenvectors of A.

(b) Give a matrix X so that

$$X^{-1}AX = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

where λ_1 and λ_2 are the eigenvalues you computed in part (a) **and** $\lambda_1 < \lambda_2$.

(c) Solve the differential equation

$$\frac{d\vec{x}}{dt} = A\vec{x} \quad \text{with} \quad \vec{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

(a) $(2-\lambda)(2-\lambda) - 1 = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3)$ $\lambda = 1, \lambda = 3$

$\lambda = 1$, $1 \cdot v_1 + v_2 = 0$ $\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{v}$ $\lambda = 3$, $-1v_1 + v_2 = 0$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b) $X = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

(c) $\vec{v}(t) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$= -1 e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

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2.(10 *points*) Assume $V \subset \mathbb{R}^n$ is a subspace

(a) Define $V^\perp$.

(b) Show that $V^\perp$ is a subspace.

(c) If V has a basis, $\{\vec{v}_1, \ldots, \vec{v}_k\}$ and $\vec{w} \perp \vec{v}_i$ for all i, show that $\vec{w}$ is in $V^\perp$.

(a) $V^\perp = \{ \vec{w} : \vec{w} \cdot \vec{v} = 0$ for all $\vec{v} \in V \}$

(b) Say $\vec{u}, \vec{w} \in V^\perp$ and so $\vec{u} \cdot \vec{v} = 0$ and $\vec{w} \cdot \vec{v} = 0$ for all $\vec{v} \in V$. Then $(\alpha \vec{u} + \beta \vec{w}) \cdot \vec{v} =$ $\alpha \vec{u} \cdot \vec{v} + \beta \vec{w} \cdot \vec{v} = 0$, for all $\vec{v} \in V$ so $\alpha \vec{u} + \beta \vec{w} \in V^\perp$ which is thus a subspace

(c) ~~For~~ ~~we~~ Any $\vec{v} \in V$ can be written as $\vec{v} = \sum \alpha_i \vec{v}_i$ by the def of a basis. Thus $\vec{w} \cdot \vec{v} = \sum \alpha_i \vec{w} \cdot \vec{v}_i = 0$ by hypothesis that $\vec{w} \cdot \vec{v}_i = 0$, Thus $\vec{w} \in V^\perp$.

3.(5 *points*) Define *orthogonal matrix* and show that if Q is orthogonal, then $(Q\vec{x}) \cdot (Q\vec{y}) = \vec{x} \cdot \vec{y}$.

Q is orthogonal if $Q^T = Q^{-1}$

Now $(Q\vec{x}) \cdot (Q\vec{y}) = (Q\vec{x})^T Q\vec{y}$

$= \vec{x}^T Q^T Q \vec{y} = \vec{x}^T Q^{-1} Q \vec{y}$

$= \vec{x}^T I \vec{y} = \vec{x}^T \vec{y} = \vec{x} \cdot \vec{y}$

Total points this page out of 15 ________

4.(10 *points*) Let

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -1 \\ 3 & 1 & 9 \end{pmatrix}$$

(a) Compute the LU decomposition of A.

(b) Using the LU decomposition, what is $\det(A)$?

(c) Using the LU decomposition, solve

$$A\vec{x} = \begin{pmatrix} 4 \\ -1 \\ 12 \end{pmatrix}$$

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & -1 \\ 3 & 1 & 9 \end{bmatrix} \xrightarrow[\to R3]{-3R1+R3} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 4 & 0 \end{bmatrix} \xrightarrow[\to R3]{-2R2+R3} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

So $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ $U = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

Solving $LU\vec{x} = \begin{pmatrix} 4 \\ -1 \\ 12 \end{pmatrix}$ first

$L\vec{y} = \begin{pmatrix} 4 \\ -1 \\ 12 \end{pmatrix}$

$y_1 = 4$

$y_2 = -1$

$3y_1 + 2y_2 + y_3 = 12$ $y_3 = 2$

Now $U\vec{x} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$

$2x_3 = 2$ $x_3 = 1$

$2x_2 - x_3 = -1$ $x_2 = 0$

$x_1 - x_2 + 3x_3 = 4$ $x_1 = 1$

$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

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5.(10 *points*) Let

$$\Phi(x_1, x_2) = x_1^4 - 2x_1^2 + x_2^2 - 10$$

(a) Find all critical points of Φ.

(b) Classify each critical point as a local max, local min, saddle, or no test.

$\nabla\Phi(x_1, x_2) = [4x_1^3 - 4x_1, 2x_2] = \vec{0}$

$0 = 4x_1^3 - 4x_1 = 4x_1(x_1^2 - 1)$

$x_1 = 0, \pm 1$

$0 = 2x_2 \quad x_2 = 0$

CRIT POINTS $(0,0), (1,0), (-1,0)$

$H\Phi(x_1, x_2) = \begin{bmatrix} 12x_1^2 - 4 & 0 \\ 0 & 2 \end{bmatrix}$

$H\Phi(0,0) = \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix}$, $\lambda = 2, -4$ (since diagonal matrix) Saddle

$H\Phi(1,0) = H\Phi(-1,0) = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$ $\lambda = 8, 2$ so both loc. min.

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