1. (15 points) Let

\[
A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}
\]

(a) Compute the eigenvalues and eigenvectors of \( A \).

(b) Give a matrix \( X \) so that

\[
X^{-1}AX = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}
\]

where \( \lambda_1 \) and \( \lambda_2 \) are the eigenvalues you computed in part (a) and \( \lambda_1 < \lambda_2 \).

(c) Solve the differential equation

\[
\frac{d\vec{x}}{dt} = A\vec{x} \quad \text{with} \quad \vec{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}
\]

\[
(9) \quad (2-\lambda)(2-\lambda) - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3) \quad \lambda_1 = 1, \lambda_2 = 3
\]

\[
\lambda = 1, \quad v_1 + v_2 = 0 \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix} = v, \quad \lambda = 3, \quad v_1 + 2v_2 = 0 \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\]

\[
(\vec{v}/4) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} e^{\lambda t} \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}^{-1} (1)
\]

\[
= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} e^{\lambda t} \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}^{-1}
\]

\[
= \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{\lambda t} + 2 e^{\lambda t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

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Total points this page out of 15
2. (10 points) Assume $V \subseteq \mathbb{R}^n$ is a subspace

(a) Define $V^\perp$.

(b) Show that $V^\perp$ is a subspace.

(c) If $V$ has a basis, $\{\vec{v}_1, \ldots, \vec{v}_k\}$ and $\vec{w} \perp \vec{v}_i$ for all $i$, show that $\vec{w}$ is in $V^\perp$.

2a) $V^\perp = \{ \vec{v} : \vec{w} \cdot \vec{v} = 0 \text{ for all } \vec{v} \in V \}$

2b) Say $\vec{u}, \vec{w} \in V^\perp$ and so $\vec{u} \cdot \vec{v} = 0$ and $\vec{w} \cdot \vec{v} = 0$ for all $\vec{v} \in V$. Then $(\alpha \vec{u} + \beta \vec{w}) \cdot \vec{v} = 0$, for all $\vec{v} \in V$ so $\alpha \vec{u} + \beta \vec{w} \in V^\perp$ which is thus a subspace.

2c) Any $\vec{v} \in V$ can be written as $\vec{v} = \sum \alpha_k \vec{v}_k$ by the definition of a basis. Thus $\vec{w} \cdot \vec{v} = \sum \alpha_k \vec{w} \cdot \vec{v}_k = 0$ by hypothesis. Thus $\vec{w} \in V^\perp$.

3. (5 points) Define orthogonal matrix and show that if $Q$ is orthogonal, then $(Q\vec{x}) \cdot (Q\vec{y}) = \vec{x} \cdot \vec{y}$.

$Q$ is orthogonal if $Q^T = Q^{-1}$

Now $(Q\vec{x}) \cdot (Q\vec{y}) = (Q\vec{x})^T Q\vec{y}

= \vec{x}^T Q^T Q\vec{y} = \vec{x}^T Q^{-1} Q\vec{y}

= \vec{x}^T \vec{y} = \vec{x} \cdot \vec{y}$
4. (10 points) Let

\[ A = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -1 \\ 3 & 1 & 9 \end{pmatrix} \]

(a) Compute the LU decomposition of \( A \).
(b) Using the LU decomposition, what is \( \text{det}(A) \)?
(c) Using the LU decomposition, solve

\[ Ax = \begin{pmatrix} 4 \\ -1 \\ 12 \end{pmatrix} \]

\[
\begin{bmatrix}
1 & -1 & 3 \\
0 & 2 & -1 \\
3 & 1 & 9 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -1 & 3 \\
0 & 2 & -1 \\
0 & 4 & 0 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -1 & 3 \\
0 & 2 & -1 \\
0 & 0 & 2 \\
\end{bmatrix}
\]

So

\[ L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix} \]

Solving

\[ LUx = \begin{pmatrix} 4 \\ -1 \\ 12 \end{pmatrix} \]

First

\[ L \bar{y} = \begin{pmatrix} \frac{4}{12} \\ -1 \\ \frac{12}{12} \end{pmatrix} \quad \bar{y}_1 = \frac{4}{12} \quad \bar{y}_2 = -1 \quad \bar{y}_3 = \frac{12}{12} \]

\[ 3\bar{y}_1 + 2\bar{y}_2 + \bar{y}_3 = 12 \quad \bar{y}_3 = 2 \]

Now

\[ U \bar{x} = \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix} \]

\[ 2\bar{x}_3 = 2 \quad \bar{x}_3 = 1 \quad 2\bar{x}_2 - \bar{x}_3 = -1 \quad \bar{x}_2 = 0 \]

\[ \bar{x}_1 - \bar{x}_2 + 3\bar{x}_3 = 4 \quad \bar{x}_1 = 1 \]

\[ \bar{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \]

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5. (10 points) Let
\[ \Phi(x_1, x_2) = x_1^4 - 2x_1^2 + x_2^2 - 10 \]

(a) Find all critical points of $\Phi$.

(b) Classify each critical point as a local max, local min, saddle, or no test.

\[ \nabla \Phi(x_1, x_2) = \begin{bmatrix} 4x_1^3 - 4x_1 & 2x_2 \end{bmatrix} = \mathbf{0} \]

\[ 0 = 4x_1^3 - 4x_1 = 4x_1(x_1^2 - 1) \]

\[ x_1 = 0, \pm 1 \]

\[ 0 = 2x_2 \quad x_2 = 0 \]

Critical points: (0, 0), (1, 0), (-1, 0)

\[ H\Phi(x_1, x_2) = \begin{bmatrix} 12x_1^2 - 4 & 0 \\ 0 & 2 \end{bmatrix} \]

\[ H\Phi(0, 0) = \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix} \quad \lambda = -4 \text{ (since diagonal matrix)} \quad \text{Saddle} \]

\[ H\Phi(1, 0) = H\Phi(-1, 0) = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix} \quad \lambda = 8, 2 \quad \text{so both local min.} \]