

(1) f is a function on $[-\pi, \pi]$, real valued

orthogonal Fourier series

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt dt, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt dt$$

Given this series, the orthonormal Fourier series is

$$f(t) \sim A_0 \left(\frac{1}{\sqrt{2\pi}} \right) + \sum_{n=1}^{\infty} A_n \frac{\cos nt}{\sqrt{\pi}} + B_n \frac{\sin nt}{\sqrt{\pi}}$$

$$A_0 = \frac{a_0}{2} \sqrt{2\pi}, \quad A_n = a_n \sqrt{\pi}, \quad B_n = b_n \sqrt{\pi}$$

The order N -truncation error sum formula is

$$\|E_N(f)\|^2 = \int_{-\pi}^{\pi} |f(t) - S_N(f)|^2 dt$$

$$= \|f\|^2 - \left[A_0^2 + \sum_{n=1}^N A_n^2 + B_n^2 \right]$$

In all answers, you need to reduce $\cos n\pi = (-1)^n$

$$a_n \approx \sin(n\pi) = 0$$

(2) f is a function on $[-\pi, \pi]$, may be complex-valued
orthonormal complex Fourier Series

$$f(t) \sim \sum_{n=-\infty}^{\infty} C_n e^{int} \quad \text{where } C_n = \int_{-\pi}^{\pi} f(t) \frac{e^{-int}}{\sqrt{2\pi}} dt$$

The order N truncation error sum formula

$$\|E_N\|^2 = \|f\|^2 - \sum_{|n| \leq N} |C_n|^2$$

The Fourier Isometry $\|f\|^2 = \sum_{n=-\infty}^{\infty} |C_n|^2$

(3) f is a real function on $[0, \pi]$

Sine Series: $f(t) \sim \sum_{n=1}^{\infty} b_n \sin nt, b_n = \frac{2}{\pi} \int_0^\pi f(t) \sin nt dt$

Cosine Series $f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt, a_n = \frac{2}{\pi} \int_0^\pi f(t) \cos nt dt$

(4) Misc Facts: $\cos n\pi = (-1)^n, \sin n\pi = 0,$

$$e^{i\theta} = \cos \theta + i \sin \theta, e^{2\pi i} = 1, e^{\pi i} = -1,$$

Integration by parts, ...

(5) The DFT
 $\vec{x} = (x_0, x_1, \dots, x_{N-1})$ is the data,
 $DFT(\vec{x}) = \vec{X}$ where $X_n = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_k w^{-kn}$

where $w = e^{2\pi i/N}$. The inverse DFT is

$IDFT(\vec{X}) = \vec{x}$ where $x_n = \sum_{k=0}^{N-1} X_k w^{kn}$

(6) In matrix form: F is a $N \times N$ matrix with
 $F_{ij} = w^{-((i-1)(j-1))}$ ($i=1, \dots, N, j=1, \dots, N$)
 and \vec{x} and $\vec{X} = DFT(\vec{x})$ are
 column vectors, then $\vec{X} = \frac{1}{\sqrt{N}} F \vec{x}$.

The inverse DFT is $\vec{x} = IDFT(\vec{X})$ with

$\vec{x} = G \vec{X}$ where $G_{ij} = w^{(i-1)(j-1)}$
 $i=1, \dots, N$
 $j=1, \dots, N$

7 If $\vec{g} = [g_0, g_1, \dots, g_{N-1}]$ and
 $h = (h_0, h_1, \dots, h_{N-1})$ their convolution is defined by
 $\vec{g} * \vec{h}(j) = \sum_{m=0}^{N-1} g_m h_{j-m}$ for $j=0, \dots, N-1$.

If H is the matrix

$$H = \begin{pmatrix} h_0 & h_{N-1} & h_{N-2} & & h_1 \\ h_1 & h_0 & h_{N-1} & \cdots & h_2 \\ \vdots & \vdots & h_0 & & \vdots \\ \vdots & & \vdots & & \vdots \\ h_{N-1} & h_{N-2} & h_{N-3} & & h_0 \end{pmatrix}$$

then $\vec{g} * \vec{h} = H\vec{g}$ when \vec{g} is treated as a column vector

8 The convolution theorem.

If $\vec{x} = \text{DFT}(\vec{x})$ and $\vec{y} = \text{DFT}(\vec{y})$

then ~~$\text{DFT}(\vec{x} * \vec{y}) = N \cdot \vec{x} * \vec{y}$~~