FS

(1) $f(t)$ is odd so all $a_n = 0$ and we compute

(a)

(b)

$$b_n = \frac{2}{\pi}\int_0^{\pi} 1 \sin(nt)\,dt = \frac{2}{\pi}\,\frac{-\cos(nt)}{n}\Big|_0^{\pi}$$

$$= \frac{2}{\pi}\left(\frac{-\cos(n\pi)+1}{n}\right) = \frac{2}{n\pi}\left((-1)^{n+1}+1\right)$$

$$\text{so } f(t) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}\left((-1)^{n+1}+1\right)\sin(nt)$$

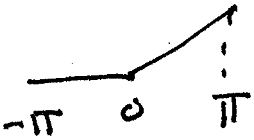
(c) $$f(t) \sim \sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\left((-1)^{n+1}+1\right)\sqrt{\pi}\right)\frac{\sin nt}{\sqrt{\pi}}$$

$$\text{So } B_n = \frac{2}{n\sqrt{\pi}}\left((-1)^{n+1}+1\right)$$

(d) $$\|f\|^2 = \int_{-\pi}^{\pi}(f(t))^2\,dt = \int_{-\pi}^{\pi} 1^2\,dt = 2\pi$$

$$\text{so } \|E_N\|^2 = 2\pi - \sum_{n=1}^{N}\frac{4}{n^2\pi}\left((-1)^{n+1}+1\right)^2$$

FS2

(a) 


(b) $$c_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt \quad (n \neq 0)$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\pi} t\, e^{-int}\, dt = \frac{1}{\sqrt{2\pi}} \left[ \frac{t\, e^{-int}}{-in} \Bigg|_0^{\pi} + \frac{1}{in} \int_0^{\pi} e^{-int}\, dt \right]$$

$$\boxed{\begin{array}{ll} \text{IBP} & \\ u = t & dv = e^{-int}\,dt \\ du = dt & v = \dfrac{e^{-int}}{-in} \end{array}}$$

$$= \frac{1}{\sqrt{2\pi}} \left[ \frac{\pi e^{-i\pi n}}{-in} + \frac{1}{in} \frac{e^{-int}}{-in} \Bigg|_0^{\pi} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[ \frac{i\pi e^{-\pi i n}}{n} + \frac{1}{n^2} \left[ e^{-in\pi} - 1 \right] \right]$$

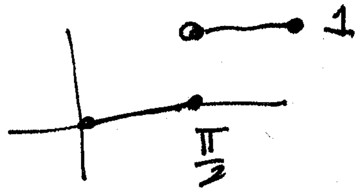
$$= \frac{1}{\sqrt{2\pi}} \left[ \frac{i\pi (-1)^n}{n} + \frac{1}{n^2} \left[ (-1)^n - 1 \right] \right]$$

$$c_0 = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t)\, dt = \frac{1}{\sqrt{2\pi}} \int_0^{\pi} t\, dt = \frac{1}{\sqrt{2\pi}} \frac{\pi^2}{2} = \frac{\pi^{3/2}}{2\sqrt{2}}$$

$$f(t) \sim \left( \frac{\pi^{3/2}}{2\sqrt{2}} \right) \left( \frac{1}{\sqrt{2\pi}} \right) + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{\sqrt{2\pi}} \left[ \frac{i\pi(-1)^n}{n} + \frac{1}{n^2} \left( (-1)^n - 1 \right) \right] \cdot \frac{e^{int}}{\sqrt{2\pi}}$$

FS3

(3) (a)




(b) $b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt = \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} \sin nt \, dt$

$= \frac{2}{n\pi}\left(-\cos nt\right)\Big|_{\frac{\pi}{2}}^{\pi} = \frac{2}{n\pi}\left(-\cos n\pi + \cos n\frac{\pi}{2}\right)$

$= \frac{2}{n\pi}\left((-1)^{n+1} + \cos n\frac{\pi}{2}\right)$

$f(t) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}\left((-1)^{n+1} + \cos\frac{n\pi}{2}\right)\sin(nt)$

(c) $a_0 = \frac{2}{\pi}\int_0^{\pi} f(t)\,dt = \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} dt = \frac{2}{\pi}\cdot\frac{\pi}{2} = 1$

$\boxed{n>0}$ $a_n = \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt \, dt = \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} \cos nt \, dt$

$= \frac{2}{\pi}\,\frac{\sin nt}{n}\Big|_{\frac{\pi}{2}}^{\pi} = \frac{2}{n\pi}\left(\sin n\pi - \sin n\pi/2\right)$

$= \frac{-2}{n\pi}\sin(n\pi/2)$, $f(t) \sim \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos(nt)$

# $\boxed{\text{DFT}}$

(1) $(1-w^q)(1+w^q+\dots+w^{(N-1)q})$

$= 1 - w^{Nq} = 1-1=0.$ since $1-w^q \neq 0$

because ~~since~~ $0<q<N$, it must be the case that

$1+w^q+\dots+w^{(N-1)q} = 0$

(2) $\text{DFT}(\vec{x} * y) = N(\mathbb{X} .* Y)$

$= 4(1\cdot 3, -1\cdot 2, 2\cdot 1, 4\cdot 7) = (12, -8, 8, 112)$

(3) $$\mathbb{X}_{N-j} = \frac{1}{N}\sum_{k=0}^{N-1} x_k w^{-(N-j)k}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} x_k w^{-Nk} w^{jk}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} \overline{x_k w^{-jk}} = \overline{\mathbb{X}_j}$$

since $\overline{x_k} = x_k$ and $\overline{w} = w^{-1}$

(4) $$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w^{-1} & w^{-2} & w^{-3} \\ 1 & w^{-2} & w^{-4} & w^{-6} \\ 1 & w^{-3} & w^{-6} & w^{-9} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w^3 & w^2 & w \\ 1 & w^2 & 1 & w^2 \\ 1 & w & w^2 & w^3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$

$w = e^{2\pi i/4} = e^{\pi i/2} = i,\ w^2 = -1,\ w^3 = -i$

DFT2

(5) $$H = \begin{bmatrix} -1 & -2 & 3 & 2 \\ 2 & -1 & -2 & 3 \\ 3 & 2 & -1 & -2 \\ -2 & 3 & 2 & -1 \end{bmatrix}$$

(6) $$X_n = \frac{1}{N} \sum_{k=0}^{N-1} 1 \cdot \omega^{kn}$$

$$= \frac{1}{N}\left[1 + \omega^n + \omega^{2n} + \dots + \omega^{(N-1)n}\right]$$

$$= \begin{cases} \frac{1}{N}(0) & \text{when } 0 < n < N \text{ (using prob (1))} \\ \frac{1}{N}(N) & \text{when } n = 0 \end{cases}$$

so $\vec{X} = (1, 0, \dots, 0)$

(7) $\cancel{e^{2\pi i k t_n} = e^{2\pi i k n/N}}$

$$\overline{e^{2\pi i(N-k)t_n}} = e^{-2\pi i(N-k)n/N} = e^{-2\pi i n}\, e^{2\pi i k n/N}$$

$$= 1\, e^{2\pi i k t_n} \quad \text{since } e^{-2\pi i n} = \left(e^{-2\pi i}\right)^n = 1^n = 1$$