(1) Q has orthonormal columns so these form an orthonormal basis for col(A) and so the projection P is given by $P = QQ^T =$

$$\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \\ 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix}$$

(2) $$B_1 = \sigma_1 \vec{u}_1 \vec{v}_1^T = 2\sqrt{6} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$$

$$= 2\sqrt{6} \begin{bmatrix} -1/\sqrt{6} & \frac{1}{\sqrt{9}} & \frac{1}{\sqrt{6}} \\ -1/\sqrt{6} & 1/\sqrt{9} & 1/\sqrt{6} \\ -1/\sqrt{6} & 1/\sqrt{9} & 1/\sqrt{6} \end{bmatrix}$$

$$= \cancel{2\sqrt{6}} \begin{bmatrix} -2 & \sqrt{6} & 2 \\ -2 & \sqrt{6} & 2 \\ -2 & \sqrt{6} & 2 \end{bmatrix}$$

$$B_2 = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T = B_1 +$$

$$\sqrt{6} \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{3} & 0 \end{bmatrix} =$$

$$= B_1 + \sqrt{6}\begin{bmatrix} 0 & -1/\sqrt{6} & 0 \\ 0 & 0 & 0 \\ 0 & 1/\sqrt{6} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & \sqrt{6} & 2 \\ -2 & \sqrt{6} & 2 \\ -2 & \sqrt{6} & 2 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -1+\sqrt{6} & 2 \\ -2 & \sqrt{6} & 2 \\ -2 & 1+\sqrt{6} & 2 \end{bmatrix}$$

(3) $$\vec{q}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 5 \\ 5 \end{bmatrix} - \left(\begin{bmatrix} -1 \\ -1 \\ 5 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}\right)\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \\ 5 \\ 5 \end{bmatrix} - 4 \cdot \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 3 \\ 3 \end{bmatrix}$$

$$\vec{q}_2 = \frac{1}{\sqrt{36}}\begin{bmatrix} -3 \\ -3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

So $Q = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$, $R = Q^T A$

$$= \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & 5 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & 6 \end{bmatrix}$$

Recall from HW that $A^+ = R^{-1}Q^T$

$$= \frac{1}{12}\begin{bmatrix} 6 & -4 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 5 & 5 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

So soln to least squares is $A^+\vec{b}$

$$= \frac{1}{12}\begin{bmatrix} 5 & 5 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8/12 \\ \frac{2}{12} \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/6 \end{bmatrix}$$

(4) $A^TA = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 12 \\ 12 & 41 \end{bmatrix}$

$\lambda = 5,\ \lambda = 45$ so $\sigma_1 = 3\sqrt{5}$ $\sigma_2 = \sqrt{5}$

$\lambda = 5$ $\vec{0} = \begin{bmatrix} 9-5 & 12 \\ 12 & 41-5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 4 & 12 \\ 12 & 36 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$\lambda = 45,\ \vec{0} = \begin{bmatrix} 9-45 & 12 \\ 12 & 41-45 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -36 & 12 \\ 12 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$ $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

so $\vec{v}_1 = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix},\ v_2 = \begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{10} \end{bmatrix}$

$V = \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ 3/\sqrt{10} & -1/\sqrt{10} \end{bmatrix},\ \Sigma = \begin{bmatrix} \sqrt{45} & 0 \\ 0 & \sqrt{5} \end{bmatrix}$

$\vec{u}_1 = \dfrac{A\vec{v}_1}{\sigma_1} = \dfrac{\begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}}{\sqrt{45}} = \begin{bmatrix} \dfrac{15}{\sqrt{450}} \\ \dfrac{15}{\sqrt{450}} \end{bmatrix} = \begin{bmatrix} 5/\sqrt{50} \\ 5/\sqrt{50} \end{bmatrix}$

$\vec{u}_2 = \dfrac{A\vec{v}_2}{\sigma_2} = \dfrac{\begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{10} \end{bmatrix}}{\sqrt{5}} = \begin{bmatrix} 5/\sqrt{50} \\ -5/\sqrt{50} \end{bmatrix}$

$U = \begin{bmatrix} 5/\sqrt{50} & 5/\sqrt{50} \\ 5/\sqrt{50} & -5/\sqrt{50} \end{bmatrix}$ $\kappa_2 = \dfrac{\sqrt{45}}{\sqrt{5}} = 3$