Your Name Printed Clearly! Key

Justify all answers! Show all work for partial credit! You will get no credit for just the answer. All problems have the same value. No calculators, no notes

EXAM 2 • BOYLAND • MAP4413 • FALL 2019

1.(10 *points*) From the definition, find the Fourier transform of

$$f(t) = e^{-2|t|}.$$

You must simplfy your answer.

$$\sqrt{2\pi}\,\hat{f}(s) = \int_{-\infty}^{\infty} e^{-2|t|}\, e^{-ist}\,dt$$

$$= \int_0^{\infty} e^{-2t}\, e^{-ist}\,dt + \int_{-\infty}^{0} e^{2t}\, e^{-ist}\,dt$$

$$= \lim_{T\to\infty} \int_0^{T} e^{(-2-is)t}\,dt + \int_{-T}^{0} e^{(2-is)t}\,dt$$

$$= \lim_{T\to\infty}\left[\frac{e^{(-2-is)t}}{-2-is}\Bigg|_0^T + \frac{e^{(2-is)t}}{2-is}\Bigg|_{-T}^{0}\right]$$

$$= \lim_{T\to\infty}\left(\frac{e^{(-2-is)T} - 1}{-2-is} + \frac{1 - e^{-(2-is)T}}{2-is}\right]$$

$$= \frac{1}{2+is} + \frac{1}{2-is} = \frac{2-is+2+is}{(2+is)(2-is)} = \frac{4}{4+s^2}$$

$$\text{so } \hat{f}(s) = \frac{1}{\sqrt{2\pi}}\,\frac{4}{4+s^2}$$

Total points this page out of 10 ______

2.(10 *points*)

$$g(t) = \frac{1}{1+t^2}$$

has Fourier transform

$$\hat{g}(s) = \frac{\sqrt{2\pi}}{2} e^{-|s|}$$

Find the Fourier transforms of

(a) $\frac{3}{1+(t-2)^2}$

(b) $\frac{1}{1+(3t-2)^2}$

(c) $\frac{-2t}{(1+t^2)^2}$ (Hint: derivative).

(a) $f(t) = 3g(t-2)$ so $\hat{f}(s) = 3\, e^{-2is} \cdot \frac{\sqrt{2\pi}}{2} e^{-|s|}$

(b) $f(t) = g(3t-2)$. To be careful, let $h(t) = g(t-2)$

so $\hat{h}(s) = e^{-2is}\hat{g}(s)$. Now $g(3t-2) = h(3t)$ so

$\mathcal{F}\, g(3t-2) = \frac{1}{3}\hat{h}\left(\frac{s}{3}\right) = \frac{1}{3} e^{-2i\frac{s}{3}} \frac{\sqrt{2\pi}}{2} e^{-|s/3|}$

(c) $f(t) = \frac{dg(t)}{dt}$ so $\hat{f}(s) = is\,\hat{g}(s)$

$= is \frac{\sqrt{2\pi}}{2} e^{-|s|}.$

Total points this page out of 10 ________

3. (10 *points*) State and prove the dilation property of the one-dimensional Fourier transform.

$$\mathcal{F}(f(at)) = \frac{1}{a}\hat{f}\left(\frac{s}{a}\right) \qquad a > 0$$

<u>Proof</u>: $$\mathcal{F}(f(at))(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(at)\, e^{-ist}\, dt$$

$$\boxed{\begin{array}{l} u = at \\ \frac{du}{a} = dt \\ \frac{u}{a} = t \end{array}}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, e^{-is\frac{u}{a}}\, \frac{du}{a}$$

$$= \frac{1}{a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, e^{-i\left(\frac{s}{a}\right)u}\, du$$

$$= \frac{1}{a}\hat{f}\left(\frac{s}{a}\right).$$

Total points this page out of 10 ______

4.(10 *points*) If $g(t)$ and $h(t)$ have Fourier series

$$g(t) = \frac{1}{2} + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{1}{4n^2-1} e^{int}$$

$$h(t) = \frac{1}{3} + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{\sin(n)}{n} e^{int}$$

Find the Fourier series of $f(x,y) = g(x)h(y)$.

$$f(x,y) \sim \frac{1}{2}\cdot\frac{1}{3} + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{3}\frac{1}{4m^2-1} e^{imx}$$

$$+ \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{2}\frac{\sin n}{n} e^{iny}$$

$$+ \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{1}{4m^2-1}\right)\left(\frac{\sin n}{n}\right) e^{i(mx+ny)}$$

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5.(10 *points*) Assume $g(t)$ and $h(t)$ have Fourier transforms

$$\hat{g}(s) = e^{-s^2} \quad \text{and} \quad \hat{h}(s) = \frac{2}{1+s^4},$$

and let $f(x,y) = g(x)h(y)$ and $p(t) = g(t) * h(t)$.

(a) Find the two-dimensional Fourier transform of f.

(b) Find the two-dimensional Fourier transform of $f(x-3, y+5)$

(c) Find the two-dimensional Fourier transform of $\frac{\partial f}{\partial y}(x,y)$

(d) Find the one-dimensional Fourier transform of p.

(a) $\hat{f}(r,s) = e^{-r^2} \frac{2}{1+s^4}$ Since separable

(b) $\mathcal{F}(f(x-3,y+5)) = e^{-3ir} e^{5is} \hat{f}(r,s)$

$= e^{-3ir} e^{5is} e^{-r^2} \frac{2}{1+s^4}$

(b) $\mathcal{F}\left(\frac{\partial f}{\partial y}\right) = is\,\hat{f}(r,s) = is\,e^{-r^2} \frac{2}{1+s^4}$

(c) $\mathcal{F}(f * g) = \hat{f} \cdot \hat{g}$ so

$\hat{p}(s) = e^{-s^2} \frac{2}{1+s^4}$

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