Row Reduction

$$\underset{A}{\begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}} \xrightarrow[-3R1+R_4 \to R_4]{\substack{-2R1+R_2 \to R_2 \\ -4R_1+R_3 \to R_3}} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix}$$

$$\xrightarrow[-4R2+R_4 \to R_4]{-3R2+R_3 \to R_3} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix} \xrightarrow{-R3+R4 \to R4} \underset{U}{\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}}$$

We will see that This implies

$$\underset{A}{\begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}} = \underset{L = \text{lower}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ \underline{2} & 1 & 0 & 0 \\ \underline{4} & 3 & 1 & 0 \\ \underline{3} & 4 & 1 & 1 \end{bmatrix}} \underset{U = \text{upper}}{\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}}$$

How do we use this to solve $A\vec{x}=\vec{b}$

Secret is that equations like $L\vec{y}=\vec{b}$ and $U\vec{x}=\vec{y}$ are easy to solve by back and forward substitution.

<u>eg</u> Solve First $L\vec{y}=\vec{b}=\begin{bmatrix}-1\\2\\3\\1\end{bmatrix}$

$$\begin{bmatrix}1&0&0&0\\2&1&0&0\\4&3&1&0\\3&4&1&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}=\begin{bmatrix}-1\\2\\3\\1\end{bmatrix}$$

as equations

$$y_1 = -1$$

$$2y_1+y_2=2 \longrightarrow y_2=2-2y_1=4$$

$$4y_1+3y_2+y_3=3 \rightarrow y_3=3-4y_1-3y_2=3+4-12=11$$

$$3y_1+4y_2+y_3+y_4=1 \rightarrow y_4=1-3y_1-4y_2-y_3=1+3-16-11=-22$$

$\boxed{\text{FORWARD SUBSTITUTION}}$

A3

Now we solve $U\vec{x} = \vec{y}$

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 11 \\ -22 \end{bmatrix}$$

$$\begin{aligned}
2x_1 + x_2 + x_3 &= -1 & & x_1 = \dots \\
x_2 + x_3 + x_4 &= 4 & \rightarrow\ & x_2 = 4 - x_3 - x_4 \\
2x_3 + 2x_4 &= 11 & \rightarrow\ & x_3 = \tfrac{1}{2}[11 - 2x_4] = \tfrac{1}{2}[11 + 22] \\
2x_4 &= -22 & \longrightarrow\ & x_4 = -11
\end{aligned}$$

$\boxed{\text{Back Substitution}}$

Then $LU(\vec{x}) = L\vec{y} = \vec{b}$ so $\vec{x}$ is sought solution