Regularity of Solutions of Hamilton-Jacobi Equation on a Domain

by

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In this lecture, we will explain a new method to show that regularity on the boundary of a domain implies regularity in the inside for PDE's of the Hamilton-Jacobi type.

We will first explain the method on a simple example that will exhibit all the ideas of the general case.

Consider a closed subset C of the Euclidean space $\mathbb{R}^N$ endowed with its usual Euclidean norm, we define the distance $d_C : \mathbb{R}^n \to [0 + \infty[$ to C by

$$d_C(x) = \inf_{c \in C} \|x - c\|.$$

Let $D \subset \mathbb{R}^N \setminus C$ be a compact smooth domain with boundary ∂D, for example D could be a closed bounded ball. If d_C is differentiable at every point of the boundary ∂D, then d_C is differentiable at every point of the interior of D.

There are several variants of this result in different settings.

One of these settings concerns continuous viscosity solutions $U : \mathbb{T}^N \times [0, +\infty[\to \mathbb{R}$ of the evolutionary equation

$$\partial_t U(x,t) + H(x, \partial_x U(x,t)) = 0,$$

where $\mathbb{T}^N = \mathbb{R}^N / \mathbb{Z}^N$, and $H : \mathbb{T}^N \times \mathbb{R}^N$ is a Tonelli Hamiltonian, i.e. $H(x,p)$ is C^2, strictly convex superlinear in p.

Let now D be a compact smooth domain with boundary ∂D contained in $T^N \times]0, +\infty[$. If U is differentiable at each point of ∂D, then this is also the case on the interior of D.