HW 11 • SPRING 2020 • PROF. BOYLAND

1. Assume $\vec{q}_1, \dots, \vec{q}_N$ is an orthonormal basis for $\mathbb{C}^N$ and $\vec{v}$ has expansion $\vec{v} = \sum_{i=1}^N \alpha_i \vec{q}_i$. The order k truncation of $\vec{v}$ is $\vec{v}^{(k)} = \sum_{i=1}^k \alpha_i \vec{q}_i$. Show that the least squares error in using $\vec{v}^{(k)}$ is

$$\|\vec{v} - \vec{v}^{(k)}\|_2^2 = \sum_{i=k+1}^{N} |\alpha_i|^2.$$

2. Let V be the vector space of all real polynomials on $[-1, 1]$ with inner product

$$\langle p, q \rangle = \int_{-1}^{1} p(t)q(t)\, dt.$$

Starting with the basis $\{1, t, t^2, \dots\}$, compute the first three orthonormal polynomials using the Gram-Schmidt process. (These are called the first three Legendre polynomials.)

3. The N^{th} root of unity is $\omega_N = e^{2\pi i/N}$.

   (a) Show that $\omega_N^N = 1$ and $|\omega_N| = 1$ and also for any integer k, $(\omega_N^k)^N = 1$, $|\omega_N^k| = 1$, and $\overline{\omega}^k = \omega^{-k}$. .

   (b) For any complex number z, show that

$$(1 - z)(1 + z + z^2 + \dots z^{N-1}) = 1 - z^N.$$

   (c) For $j = 0, \dots, N-1$, let $\vec{q}_j$ be defined by

$$\vec{q}_j = \frac{1}{\sqrt{N}}[1, \omega_N^j, \omega_N^{2j}, \dots, \omega_N^{(N-1)j}]^T.$$

   So, for example,

$$\vec{q}_0 = \frac{1}{\sqrt{N}}[1, 1, 1, \dots, 1]^T.$$

   and

$$\vec{q}_3 = \frac{1}{\sqrt{N}}[1, \omega_N^3, \omega_N^6, \dots, \omega_N^{(N-1)3}]^T.$$

   Show using parts (a) and (b) that $\{\vec{q}_0, \vec{q}_1, \dots, \vec{q}_{N-1}\}$ is an orthonormal basis for $\mathbb{C}^N$. Note: the index of this basis is traditionally $0, 1, \dots, N-1$ to keep the formulas simpler.

4. Let f on $[-\pi, \pi]$ be given by

$$f(t) = \begin{cases} 1 & \text{if } |t| < \pi/4 \\ 0 & \text{if } |t| \geq \pi/4 \end{cases}$$

Compute (by hand) the series expansion in $L^2([-\pi, \pi])$ of f with respect to the Fourier orthonormal basis

$$\{\frac{e^{int}}{\sqrt{2\pi}}\} \text{ with } n = 0, \pm 1, \pm 2, \dots$$