#HW4

(1) $A = \begin{bmatrix} -5 & -3 \\ 6 & 4 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$,

$X = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$, we need $X^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

So MATRIX Form:

$$\vec{x}(t) = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Vector Form

$$\vec{x}(t) = -e^{t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(2) $\begin{vmatrix} A_{11} - \lambda & A_{12} \\ A_{21} & A_{22} - \lambda \end{vmatrix} = (A_{11} - \lambda)(A_{22} - \lambda) - A_{12} A_{21}$

$= \lambda^2 - (A_{11} + A_{22})\lambda + A_{11} A_{22} - A_{12} A_{21}$

$= \lambda^2 - T\lambda + D \Rightarrow$ quad formula.

To get the idea let $n=1$,

$$\Phi(x) = bx + \frac{1}{2} x S x = bx + \frac{1}{2} S x^2$$

so $\nabla \Phi(x) = b + Sx$

$H\Phi(x) = S.$

Now $n=2$

$$\Phi(x_1, x_2) = [b_1\, b_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{1}{2} [x_1\, x_2] \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= b_1 x_1 + b_2 x_2 + \frac{1}{2} \left[S_{11} x_1^2 + 2 S_{12} x_1 x_2 + S_{22} x_2^2 \right]$$

so $\nabla \Phi(x_1, x_2) = [b_1 + S_{11} x_1 + S_{12} x_2,\ b_2 + S_{12} x_1 + S_{22} x_2]$

$$= \vec{b}^T + [x_1, x_2] \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix}$$

$$= \vec{b}^T + \vec{x}^T S$$

and $H\Phi(x_1, x_2) = \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix} = S$

which should be the same in general.

For general n

$$\Phi(x) = \sum_{i=1}^{n} b_i x_i + \frac{1}{2}\left[\sum S_{ii} x_i^2 + 2\sum_{i \neq j} S_{ij} x_i x_j\right]$$

So $$(\nabla \Phi(x))_i = \frac{\partial \Phi}{\partial x_i} = b_i + S_{ii} x_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} S_{ij} x_j$$

$$= b_i + \vec{x}^T \cdot [S_{i1}, S_{i2}, \ldots, S_{in}]$$

So as a matrix $\nabla \Phi(x) = \vec{b} + \vec{x}^T S$

and $$(H\Phi(x))_{ij} = \frac{\partial^2 \Phi}{\partial x_i \partial x_j} = S_{ij}$$

So as a matrix

$$H\Phi(x) = S.$$