#59

(1) $A = U\Sigma V^T$ is full SVD and we don't assume A is square. Let's say $m \geq n$ for concreteness. So pictorially

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} U \end{bmatrix} \underbrace{\begin{bmatrix} \ddots \end{bmatrix}}_{\Sigma} \begin{bmatrix} V^T \end{bmatrix}$$

where U and V are square and orthogonal

(a) $A^T = V\Sigma^T U^T$ will be SVD for A^T and note that if $m > n \Rightarrow \Sigma^T \neq \Sigma$.

(b) $A^T A = V\Sigma^T U^T U \Sigma V^T = V \Sigma^T \Sigma V^T$

In the case $m \geq n$, $\Sigma^T\Sigma = \underbrace{\begin{bmatrix} \ddots & \end{bmatrix}}_{n \times m} \underbrace{\begin{bmatrix} \ddots \\ \\ \end{bmatrix}}_{m \times n}$

$$= \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix}$$

(c) Assuming A is invertible, so $\sigma_n > 0$ and $n \times n$

$$A^{-1} = (V^T)^{-1}\Sigma^{-1}U^{-1} = V\Sigma^{-1}U^T$$

with $\Sigma^{-1} = \begin{bmatrix} \sigma_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \sigma_n^{-1} \end{bmatrix}$

$\boxed{\#5b}$

(2) $AA^T = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix}$ has char poly

$p(\lambda) = -\lambda(\lambda^2 - 6\lambda + 6)$ so $\lambda = 0,\ 3 \pm \sqrt{3}$

and nonzero sing values are $\sigma_1 = \sqrt{3+\sqrt{3}}$, $\sigma_2 = \sqrt{3-\sqrt{3}}$

(3) $A^TA = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}$ so $\lambda = 2, 8$, $\sigma_1 = \sqrt{8}$ $\sigma_2 = \sqrt{2}$ $\rightarrow$ $\boxed{\text{e.vect are } \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \vec{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

$$\vec{u}_1 = \frac{A\vec{v}_1}{\sigma_1} = \frac{1}{2\sqrt{2}}\begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$\vec{u}_2 = \frac{A\vec{u}_2}{\sigma_2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

so $\Sigma = \begin{bmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$ $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$U = \begin{bmatrix} 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{bmatrix}$$