HW 7 • SPRING 2020 • PROF. BOYLAND

1. Let A be a $(m \times n)$-matrix.

   (a) Show that $\|A\|_F^2 = \text{trace}(A^T A) = \text{trace}(AA^T)$.

   (b) If U and V are $(m \times m)$ and $(n \times n)$ orthogonal matrices, respectively, show that $\|UA\|_F = \|AV\|_F = \|A\|_F$.

   (c) If A has full SVD, $A = U\Sigma V^T$, with r nonzero singular values, show that $\|A\|_F^2 = \sigma_1^2 + \cdots + \sigma_r^2$.

2. In this exercise you will investigate the conditioning of multiplication by an ill-conditioned matrix. Your answer must include any code and the results of running it. You should use built-in or library functions of your system.

   (a) In what follows, $\vec{x}$ is the base vector, $\vec{y}$ is the perturbation direction, and $\epsilon > 0$ is the size of the perturbation and so the perturbed vector is $\vec{z} = \vec{x} + \epsilon\vec{y}$. In this case the specific condition number of matrix multiplication $f(\vec{x}) = A\vec{x}$ associated with $\vec{x}, \vec{y}$ and ϵ is

$$\kappa(\vec{x}, \vec{y}, \epsilon) = \frac{\frac{\|\delta f\|}{\|f\|}}{\frac{\|\delta \vec{x}\|}{\|\vec{x}\|}} = \frac{\frac{\|A\vec{z} - A\vec{x})\|}{\|A\vec{x}\|}}{\frac{\|\vec{z} - \vec{x}\|}{\|\vec{x}\|}} \tag{1}$$

   Now assume all norms are two-norms and that $\vec{x}$ and $\vec{y}$ are unit vectors. Show that in this case the formula (**??**) reduces to

$$\kappa(\vec{x}, \vec{y}, \epsilon) = \frac{\|A\vec{y}\|}{\|A\vec{x}\|} \tag{2}$$

   Notice that the ϵ has dropped out, so it is independent of the size of the perturbation. This is because it is a *relative* condition number. We now just write $\kappa(\vec{x}, \vec{y})$.

   (b) Create the 5×5 Hilbert matrix H via the rule

$$H_{i,j} = \frac{1}{i + j - 1}$$

   .

   (c) Compute the SVD, $H = U\Sigma V^T$.

   (d) Compute the usual, two-norm condition number of H. It should be approx 4.8×10^5.

   (e) Now create $\vec{x}$ and $\vec{y}$ as random, (5×1) unit vectors and compute $\kappa(\vec{x}, \vec{y})$. Repeat this 10 times and compute the max, min, mean and standard deviation of your results.

   (f) Now let $\vec{x} = \vec{v}_5$ the last, right singular vector and $\vec{y} = \vec{v}_1$ the first, right singular vector and compute $\kappa(\vec{x}, \vec{y})$.

   (g) Explain the disparity and relationship between the results in parts (d), (e) and (f) using equation (**??**).