1a) Let $A = [\vec{a}_1 \dots \vec{a}_n]$ with $A_j = \begin{bmatrix} A_{1j} \\ A_{2j} \\ \vdots \\ A_{mj} \end{bmatrix}$

then $(A^TA)_{jj} = \vec{a}_j^T \vec{a}_j$

$= [A_{1j} \dots A_{mj}] \begin{bmatrix} A_{1j} \\ \vdots \\ A_{mj} \end{bmatrix} = A_{1j}^2 + \dots + A_{mj}^2 = \sum_{i=1}^{m} A_{ij}^2$

Then trace $(A^TA) = \sum_{j=1}^{n} A_{jj} = \sum_{j=1}^{n}\sum_{i=1}^{m} A_{ij}^2 = \|A\|_F^2$

The other case is similar

(b) $\|AU\|_2^2 = \text{trace}((AU)(AU^T)) = \text{trace } AUU^TA^T$ using part (a) and U orthogonal

$= \text{trace } AA^T = \|A\|_2^2$

The other case is similar

(c) If $A = U\Sigma V^T$ full SVD with U and V orthogonal, using (b)

$\|A\|_F^2 = \|U\Sigma V^T\|_F^2 = \|\Sigma\|_F^2 = \sigma_1^2 + \dots + \sigma_r^2$

Since these are the only nonzero elements in Σ.

(2a) $A\vec{z} - \cancel{Ax} = A(\vec{x} + \varepsilon\vec{y}) - A\vec{x} = A\vec{x} + A\varepsilon\vec{y} - Ax$
$= \varepsilon(A\vec{y})$ and $z - x = \vec{x} + \varepsilon\vec{y} - \vec{x} = \varepsilon\vec{y}$

Thus
$$\frac{\dfrac{\|A\vec{z} - A\vec{x}\|}{\|A\vec{x}\|}}{\dfrac{\|\vec{z} - \vec{x}\|}{\|\vec{x}\|}} = \frac{\dfrac{\|\varepsilon A\vec{y}\|}{\|A\vec{x}\|}}{\dfrac{\|\varepsilon\vec{y}\|}{\|x\|}} = \frac{\varepsilon\dfrac{\|A\vec{y}\|}{\|A\vec{x}\|}}{\varepsilon\dfrac{\|\vec{y}\|}{\|x\|}}$$

$$= \frac{\|A\vec{y}\|}{\|A\vec{x}\|}$$ since $1 = \|\vec{x}\| = \|\vec{y}\|$

2g: For random $\vec{y}$ and $\vec{x}$ one expects that $\|A\vec{y}\| \approx \|A\vec{x}\|$ so $K(\vec{x}, \vec{y})$ typically around 1.

But $\vec{v}_1$ is the direction which gets stretched the most by A, $A\vec{v}_1 = \sigma_1 \vec{u}_1$ so $\|A\vec{v}_1\| = \|\sigma_1 \vec{u}_1\| = \sigma_1$. And $\vec{v}_5$ is the direction in which A shrinks the most, $A\vec{v}_5 = \sigma_5 \vec{v}_5$ and so $\|A\vec{v}_5\| = \|\sigma_5 \vec{u}_1\| = \sigma_5$

and so $K(\vec{x}, \vec{y}) = \sigma_5/\sigma_1$, the condition number.
so the worst case is realized by $\vec{y} = \vec{v}_1$ and $\vec{x} = \vec{v}_5$.