(1) The important fact is that, if Q has orthonormal columns, then $Q^TQ=I$ even when Q is not square

(a) $A=\hat{U}\hat{\Sigma}V^T \Rightarrow A^+=(A^TA)^{-1}A^T$

$$(V\hat{\Sigma}^T\hat{U}^T\hat{U}\hat{\Sigma}V^T)^{-1}V\hat{\Sigma}^T\hat{U}^T$$

$$= (V\hat{\Sigma}^2V^T)^{-1}V\hat{\Sigma}^T\hat{U}^T$$

$$= V\hat{\Sigma}^{-2}V^TV\hat{\Sigma}\hat{U}^T$$

$$= V\hat{\Sigma}^{-2}\hat{\Sigma}\hat{U}^T = V\hat{\Sigma}^{-2}\hat{U}^T$$

$$=$$

using the fact that $\hat{\Sigma}$ is diagonal with all nonzero entries on the diag and $V^T=V^{-1}$ since V is orthogonal

(b) $A=\hat{Q}\hat{R}$, $A^+=(A^TA)^{-1}A^T$

$$(\hat{R}^T\hat{Q}^T\hat{Q}\hat{R})^{-1}\hat{R}^T\hat{Q}^T$$

$$= (\hat{R}^T\hat{R})^{-1}\hat{R}^T\hat{Q}^T = \hat{R}^{-1}\hat{R}^{-T}\hat{R}^T\hat{Q}^T$$

$$=$$

$$= \hat{R}^{-1}\hat{Q}^T$$ using the fact that $\hat{R}$ is invertible since all its diagonal elements are not zero.

(2) $\vec{q}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} / \sqrt{25} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$

$$\vec{v}_2 = \begin{bmatrix} 5 \\ -5 \end{bmatrix} - \left(\begin{bmatrix} 5 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} \right) \cdot \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$$

$$= \begin{bmatrix} 28/5 \\ -21/5 \end{bmatrix}, \quad \|\vec{v}_2\| = 7 \text{ so } \vec{q}_2 = \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}$$

$$\text{so } Q = \begin{bmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{bmatrix} \text{ and } A = QR$$

$$\text{so } R = Q^T A = \begin{bmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 4 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 \\ 0 & 7 \end{bmatrix}$$

(3) Both methods should yield the same result

$$x \approx \begin{bmatrix} 0.93 \\ 1.72 \\ 0.138 \\ 1.596 \\ 2.468 \end{bmatrix}$$