Guidelines: You must do seven (7) problems for you final homework worth 20% of the total grade. You can do any additional problems for extra credit up to 10% of the total grade. Due Thursday, December 15. It needs to be hard copy and dropped off at my office (slid under door if I am not there). You can turn it in earlier if you want.

General notation: The Hausdorff distance between two sets is written $d_h(A, B)$. The box or fractal dimension of a set (this is the first dimension we did, so it is *not* the Hausdorff dimension) is written $dim_B(K)$. The metric on the sequence space Σ_2^+ is

$$\rho(\underline{s}, \underline{s}') = \sum_{i=0}^{\infty} \frac{|s_i - s_i'|}{2^i}.$$

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1. For a sequence $\underline{s} = .s_0, s_1, \cdots \in \Sigma_2^+$ define $\alpha(\underline{s}) = .1s_0, s_1, \ldots$.
   (a) Show that $\alpha : \Sigma_2^+ \to \Sigma_2^+$ is a contraction.
   (b) Find the contraction constant.
   (c) Find the fixed point of α.
2. Give two sets A, B in the plane with $A \subset B$ and $d_H(A, B) = 1$.
3. Let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ be defined by $f_1(x) = (1/2)x$ and $f_1(x) = (1/2)x + (1/2)$
   (a) Find the set K so that $K = f_1(K) \cup f_2(K)$.
   (b) Find the dimension $dim_B(K)$.
4. Show that $f : (X, d) \to (Y, \rho)$ is continuous if and only if for any sequence $\{x_n\} \subset X$, $x_n \to x$ implies $f(x_n) \to f(x)$.
5. Let Δ_1 be the triangle with vertices $(0, 0), (1, 0)$ and $(0, 1)$ and Δ_2 be the triangle with vertices $(0, -1), (-1/2, -1)$ and $(0, -3/2)$. Find a similarity f with $f(\Delta_1) = \Delta_2$.
6. If (X, d) is a metric space, fix a point $x_0 \in X$ and define a function $f : X \to \mathbb{R}$ by $f(x) = d(x, x_0)$. Show that f is continuous.
7. Define $f_1, f_2, f_3 : \mathbb{R} \to \mathbb{R}$ as $f_1(z) = (1/5)z$, $f_2(z) = (1/5)z + (4/5, 0)$ and $f_3(z) = -(1/5)z + (0, 4/5)$, and for a compact set Z, let $H(Z) = f_1(Z) \cup f_2(Z) \cup f_3(Z)$
   (a) If T is the triangle with vertices $(0, 0), (1, 0), (0, 1)$, draw a picture of $H(T)$ and $H^2(T)$.
   (b) Let K be the fractal with $H(K) = K$. Find the dimension $dim_B(K)$.
8. Let a be a number with $0 < a < 1$. The middle a-Cantor set C_a is constructed as follows. First remove the middle a^{th} interval from $[0, 1]$ (so you are removing the open interval

$(.5-a/2, .5+a/2))$. Now remove the middle a^{th} interval from the remaining two intervals, and repeat the process infinitely. The set that remains is C_a.

(a) Define two similarities $f_1, f_2 : [0,1] \to [0,1]$ (which will depend on a) so that $C_a = f_1(C_a) \cup f_2(C_a)$.

(b) Compute the dimension $dim_B(C_a)$.

(c) Show that there is no bi-Lipschitz transformation α with $\alpha(C_a) = C_{a'}$ when $a \neq a'$.

9. Let $f_1, \ldots, f_n$ all be similarities with ratio r and S be the unit square. If for all $i \neq j$, $f_i(S) \cap f_j(S) = \emptyset$ and $K = \cup f_i(K)$, show that $dim_B(K) = -\ln(n)/\ln(r)$.

10. Let C be the usual middle third Cantor set and $C' = [0, 1/3] \cap C$. Show that $dim_B(C) = dim_B(C')$.

11. Let C be the standard middle third Cantor set and $f : [0,1] \to [1,2]$ is defined by $f(x) = 2^x$. Let

$$C' = f(C) = \{f(x) \colon x \in C\}.$$

Compute the Hausdorff dimension of C' and be sure to justify your answer thoroughly.

12. The figure below shows the construction of the Menger sponge. Compute its fractal dimension.

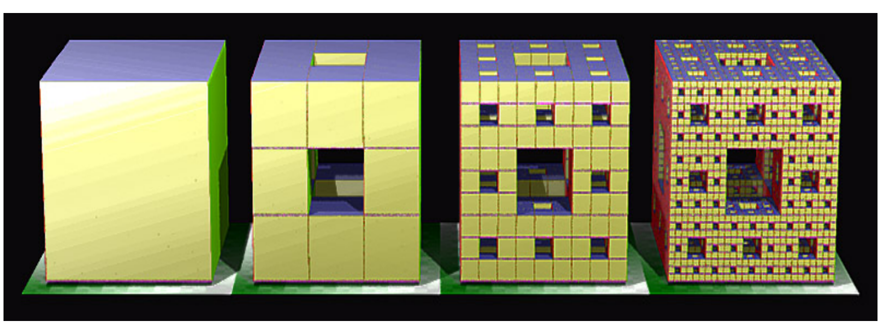

FIGURE 1. First three steps in construction of Menger sponge