Isotopy stable dynamics relative to compact invariant sets

Philip Boyland
Department of Mathematics
University of Florida
PO Box 118105
Gainesville, FL 32611-8105
boyland@math.ufl.edu

Toby Hall
Department of Mathematical Sciences
University of Liverpool
Liverpool L69 3BX
United Kingdom
tobyhall@liv.ac.uk

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Abstract

Let f be an orientation-preserving homeomorphism of a compact orientable manifold. Sufficient conditions are given for the persistence of a collection of periodic points under isotopy of f relative to a compact invariant set A. Two main applications are described. In the first, A is the closure of a single discrete orbit of f, and f has a Smale horseshoe, all of whose periodic orbits persist; in the second, A is a minimal invariant Cantor set obtained as the limit of a sequence of nested periodic orbits, all of which are shown to persist under isotopy relative to A.

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1 Introduction

Dynamical properties which are exhibited by every homeomorphism in an isotopy class are said to be isotopy stable. The identification of isotopy stable features makes it possible to obtain a great deal of information about the dynamics of a map given only limited algebraic or combinatorial data concerning its isotopy class. The simplest example of this is provided by the Lefschetz fixed point theorem whose conclusion, from this point of view, is the existence of an isotopy stable fixed point. Nielsen fixed point theory yields a refinement of the Lefschetz theorem, by means of which it may be possible to prove the isotopy stability of a collection of several fixed points. Somewhat surprisingly, the application of this theory to periodic points is comparatively recent, with the work of Halpern [18], Jiang [23] and others. The proceedings [28] give a good picture of the current state of the theory.

Unremovability is the strongest sense in which a periodic point can be isotopy stable. It arises from the bifurcation-theoretic notion of following a periodic point through a continuously varying family of homeomorphisms: if f_t is such a family (i.e. an isotopy), and $t \mapsto p_t$ is a path in M with the property that p_t is a periodic point of f_t with period n

for all t, then the pairs (p_0, f_0) and (p_1, f_1) are said to be *strong Nielsen equivalent*. The pair (p, f) is said to be *unremovable* if every g isotopic to f has a periodic point q such that (p, f) and (q, g) are strong Nielsen equivalent. In other words, any g isotopic to f has a periodic point q which shares with p any dynamical property which is preserved as it is followed continuously through a family of homeomorphisms.

Using bifurcation-theoretic techniques, sufficient conditions for the unremovability of a periodic point of a C^1-diffeomorphism were given by Asimov and Franks [1]. Their work was subsequently generalized, using more traditional Nielsen-theoretic methods, to collections of periodic points of homeomorphisms under isotopy relative to a finite invariant set [16]. By means of the Nielsen-Thurston classification theorem, necessary and sufficient conditions have been given for the unremovability of fixed points [24] and periodic points [7] of surface homeomorphisms. In this paper, sufficient conditions are given for the unremovability of collections of periodic points under isotopy relative to a general compact invariant set. The main application anticipated for these results is to surface homeomorphisms, but the results are valid for manifolds of any dimension greater than one.

Nielsen theory relative to an invariant set A has been considered by several authors, and is used to find lower bounds on the number of periodic orbits in the whole space, the invariant set, and the complement of the invariant set. [31] is a good survey article for the interested reader. In contrast to the approach of such papers, there is no requirement here that A be an Absolute Neighborhood Retract. In fact, A is a Cantor set in many of the more substantial applications. This level of generality is achieved by finding conditions which imply that Nielsen classes remain bounded away from A.

The bifurcation-theoretic approach provides a useful heuristic for understanding the conditions for unremovability which are presented here. Given a periodic point p of a homeomorphism f, one can ask how the periodic point might disappear as it is followed through a family of homeomorphisms. There are three obvious scenarios under which this can occur. In the first, the continuation of p collides with the continuation of a periodic point on a different orbit, and the two annihilate each other (saddle-node bifurcation). For this to happen two conditions must be satisfied. First, the two periodic points must be able to collide and so they must lie in the same strong Nielsen equivalence class (that is, the topological manner in which they are embedded in $M \setminus A$ must be the same). Second, in order that they can annihilate each other, the sum of their periodic point indices must be zero. Thus this scenario can be ruled out by requiring that the set of all periodic points of f which are strong Nielsen equivalent to p has non-zero index. This condition is expressed by saying that p has an *essential* strong Nielsen class.

The second scenario is that the continuation of p collides with the continuation of another point on the same orbit (period-dividing bifurcation). This is ruled out by the *uncollapsibility* hypothesis, a precise topological statement that the periodic point never period divides. This high-level condition is used in the statement of the theorem because there are a number of different hypotheses, often verifiable in practice, which imply uncollapsibility. Some of these are described in Section 3. Notice that a fixed point cannot period divide, and hence this condition only comes into play when considering periodic points of period greater than one.

The final scenario is that the continuation of p runs into A. If A is topologically simple, then p will run into a periodic point in A which is strong Nielsen equivalent to p, and in such cases p should be considered to be unremovable. In cases where the topology of A and the dynamics of f restricted to A are sufficiently complicated, there are other possibilities to consider: the continuation of p could limit to a non-trivial continuum in A, or to a periodic point to which it is not strong Nielsen equivalent. Disallowing these possibilities requires a new Nielsen-theoretic ingredient, *strong Nielsen boundedness*. Conditions which imply strong Nielsen boundedness are given in Section 3.

The main theorem (proved in Section 2.3) is:

Theorem 2.4 *For an orientation preserving homeomorphism of a compact smooth orientable manifold M with compact invariant set A, a periodic point collection which is uncollapsible, strong Nielsen bounded, and essential* rel A *is unremovable* rel A.

Notice that the theorem is stated for (finite) collections of periodic points, rather than single periodic points as in the above discussion. There are two reasons for working with periodic point collections. First, the persistence of a collection means more than that the individual orbits persist: it also implies the persistence of the topological relationship between those orbits. This is often useful in applications. Second, proving the unremovability of a collection of periodic points is sometimes easier than proving their individual unremovability, since the topological relationship between the orbits can present additional obstructions to removability (see Remark 3.2 d)).

As noted above, the main contributions of this paper are techniques for doing Nielsen theory relative to topologically complicated invariant sets A. The principal new feature in this case is the third scenario discussed above. In finding conditions that disallow this scenario it does not suffice to require that the given periodic point is not Nielsen equivalent to any points in A (see the example in Section 4.2). Rather, one must restrict attention to the Nielsen bounded classes, i.e. the classes that are inside the complement of A for any isotopic map. In finding conditions that imply Nielsen boundedness it is necessary that the dynamics on A be known: one cannot hope to do Nielsen theory on noncompact spaces without information about the action on the "ends". The action on the ends is clear when the problem is stated in terms of isotopies relative to a compact invariant set. Working relative to an invariant set has the added advantage of making the the fixed point index straightforward; the index of a fixed point rel A is just the index in M. However, this point of view has its own complications. Contrary to the situation in the absence of a selected invariant set, if A contains accumulation points then two rel A homeomorphisms can be arbitrarily close (as homeomorphisms of M) and still not be isotopic rel A.

Working in the noncompact space A^c does have one very convenient feature. If two periodic points are said to be Nielsen equivalent rel A if they are Nielsen equivalent in A^c, then most of the standard aspects of Nielsen theory (excluding the index) go through with little change. A familiarity with this standard Nielsen theory will be assumed: [23] is an excellent reference.

Section 2 consists of definitions, basic lemmas and the proof of the main theorem, which is based on the approach of [16]. Conditions which imply Nielsen boundedness and uncollapsibility are then described in Section 3. Finally, a number of examples and applications of the main theorem are presented in Section 4. Those of Sections 4.1 and 4.2 are relatively simple, and are intended to illustrate some of the definitions and the application of the main theorem; more substantial applications are given in Sections 4.3 and 4.4.

2 The Unremovability theorem

2.1 Definitions

Let M be a compact smooth orientable manifold with metric d, and A be a compact subset of M. The complement of A in M is denoted $A^c = M \setminus A$. Let $\mathrm{Aut}(M, A)$ denote the space of orientation preserving homeomorphisms $f\colon (M, A) \to (M, A)$ equipped with the uniform metric ρ induced by d. For each integer $n \geq 1$, let $P_n(f)$ be the set of (least) period n points of f: thus $P_n(f)$ is an open subset of $\mathrm{Fix}(f^n)$. The orbit of a point $p \in M$ under f is written $o(p, f) = \{f^i(p) : i \in \mathbb{Z}\}$.

In this paper, collections of $k \geq 1$ distinct periodic points will be considered: their periods will be encoded in a vector $\mathbf{n} = (n_1, \ldots, n_k)$. The reader may find it helpful to concentrate initially on the case $k = 1$. Write $\mathcal{F}_{\mathbf{n}}(f)\colon M^k \to M^k$ for the map $f^{n_1} \times \cdots \times f^{n_k}$, and define $P_{\mathbf{n}}(f) \subseteq M^k$ for the set of points $\mathbf{p} = (p_1, \ldots, p_k)$ such that $p_\alpha \in P_{n_\alpha}(f)$ for each $\alpha \leq k$, and the points p_α all lie on different orbits of f: clearly $P_{\mathbf{n}}(f)$ is an open subset of $\mathrm{Fix}(\mathcal{F}_{\mathbf{n}}(f))$. In practice, whenever k and $\mathbf{n}$ are either arbitrary or fixed without ambiguity, the subscripts will be dropped and the symbols $\mathcal{F}(f)$ and $P(f)$ used. Moreover, the symbols α and β will always be taken to range over $\{1, \ldots, k\}$, and expressions such as 'for some α' or 'for all β' should be interpreted accordingly. The product metric on M^k will also be denoted d. Finally, in an abuse of notation, $\mathbf{p} \in A^c$ will be taken to mean that $p_\alpha \notin A$ for all α, and $\mathbf{p} \notin A^c$ to mean the converse, that $p_\alpha \in A$ for some α.

If $f \in \mathrm{Aut}(M, A)$ and $\mathbf{p} \in P(f)$, then the pair $(\mathbf{p}, f) \in M^k \times \mathrm{Aut}(M, A)$ is referred to as a *periodic point collection*: the set of all periodic point collections with periods $\mathbf{n}$ is denoted $\mathrm{PPC}(\mathbf{n})$ (or just PPC):

$$\mathrm{PPC} = \{(\mathbf{p}, f) \in M^k \times \mathrm{Aut}(M, A) : \mathbf{p} \in P(f)\}.$$

Similarly FIX($\mathbf{n}$) (or just FIX) is defined as:

$$\mathrm{FIX} = \{(\mathbf{p}\,, f) \in M^k \times \mathrm{Aut}(M, A) : \mathbf{p} \in \mathrm{Fix}(\mathcal{F}(f))\},$$

and thus PPC is an open subset of FIX.

Isotopies (respectively homotopies) will be written in the form $f_t\colon f \simeq g$. This means that there are homeomorphisms (respectively maps) $f_t\colon M \to M$ for each $t \in [0,1]$, such that the map $M \times I \to M$ given by $(p,t) \mapsto f_t(p)$ is an isotopy (respectively homotopy) from f to g. Isotopies and homotopies will usually be taken relative to A: in this case, all the maps f_t are required to agree *pointwise* on A. Isotopies and homotopies of arcs and paths relative to A are defined similarly: such an arc or path is allowed to pass through A, but intersections with A must remain fixed throughout the isotopy or homotopy. To be more precise, if $\alpha\colon [0,1] \to M$ is a path, and $\alpha(t^*) \in A$ for some $t^* \in [0,1]$, then $\beta(t^*) = \alpha(t^*)$ for any path β which is homotopic to α rel A. A homotopy $\{\mathcal{F}_t\}\colon \mathcal{F}(f) \simeq \mathcal{F}(g)$ is said to be relative to A if

$$\pi_i(\mathcal{F}_t(x_1, \ \ldots\ , x_{i-1}, a, x_{i+1}, \ \ldots\ , x_k)) = f^{n_i}(a) = g^{n_i}(a)$$

for $a \in A$ and all $1 \leq i \leq k$ where $\pi_i : M^k \to M$ is the projection onto the i^{th} component. Homotopy of paths in M^k relative to A is defined similarly.

Two fixed points $p, q \in A^c$ of an iterate f^n of f are said to lie in the same f^n-Nielsen class rel A if they lie in the same Nielsen class for the restriction of f^n to A^c in the usual sense:

Definition 2.1 Suppose that $p, q \in \mathrm{Fix}(f^n) \cap A^c$ for some integer $n \geq 1$. Then p and q *lie in the same f^n-Nielsen class* rel A if there is a path α in A^c from p to q such that α and $f^n \circ \alpha$ are homotopic relative to endpoints in A^c.

By contrast, the definitions of two periodic point collections being Nielsen equivalent or strong Nielsen equivalent rel A involve working rel A rather than in A^c. In particular, this means that the definitions are valid even if some of the points in the collection lie in A.

Definitions 2.2 Two elements $(\mathbf{p}\,, f)$ and $(\mathbf{q}\,, g)$ of FIX are *Nielsen equivalent* rel A if there exists a path α in M^k from $\mathbf{p}$ to $\mathbf{q}$, and a homotopy $\mathcal{F}_t\colon \mathcal{F}(f) \simeq \mathcal{F}(g)$ relative to A such that the path $t \mapsto \mathcal{F}_t(\alpha(t))$ is homotopic (with fixed endpoints) to α rel A.

Two elements $(\mathbf{p}\,, f)$ and $(\mathbf{q}\,, g)$ of PPC are *Strong Nielsen equivalent* rel A, written $(\mathbf{p}\,, f) \sim (\mathbf{q}\,, g)$, if there exists a path γ in M^k from $\mathbf{p}$ to $\mathbf{q}$, and a rel A isotopy $f_t\colon f \simeq g$ with $\gamma(t) \in P(f_t)$ for all t. It will be said that $(\mathbf{p}\,, f) \sim (\mathbf{q}\,, g)$ *by the path γ and the isotopy f_t*. The *Strong Nielsen Class* of an element $(\mathbf{p}\,, f) \in$ PPC is defined to be

$$\mathrm{snc}(\mathbf{p}\,, f) = \{\mathbf{q} \in P(f) : (\mathbf{q}\,, f) \sim (\mathbf{p}\,, f)\}.$$

Remarks 2.1

a) Notice that in Definition 2.2 there is no requirement that the periodic points or the paths lie in A^c.

b) The 'rel A' will be dropped where there is no danger of confusion.

c) It is clear that Nielsen equivalence and Strong Nielsen equivalence are equivalence relations on FIX and PPC respectively. The relation of Nielsen equivalence corresponds to what is normally termed correspondence of Nielsen classes under a homotopy (see [23]). The equivalence classes under strong Nielsen equivalence are just the path components of PPC.

d) It can easily be seen that Strong Nielsen equivalence implies Nielsen equivalence: simply choose $\alpha = \gamma$ and $\mathcal{F}_t = \mathcal{F}(f_t)$.

e) If f is a fixed homeomorphism, then it would seem natural to refer to the set S of all $\mathbf{q}$ in $P(f)$ for which $(\mathbf{q}\,, f)$ is Nielsen equivalent to $(\mathbf{p}\,, f)$ as the Nielsen class of $(\mathbf{p}\,, f)$. This is not correct because it is not the $\mathcal{F}(f)$-Nielsen class of $\mathbf{p}$ in the usual sense. This remains the case even when each $\mathbf{q} \in S$ can be shown to be Nielsen equivalent to $(\mathbf{p}\,, f)$ using a path α in A^c: what S is in this case is the union of all $\mathcal{F}(f)$-Nielsen classes which correspond to that of $\mathbf{p}$ under a homotopy. The set S is the usual Nielsen class if every self-homotopy of $\mathcal{F}(f)$ lifts to a self-homotopy in the universal cover of $(M \setminus A)^k$: that is, if the Jiang subgroup of $\mathcal{F}(f)$ is trivial [23]. This is always the case if the center of the fundamental group of $(M \setminus A)^k$ is trivial.

If $(\mathbf{p}\,, f) \sim (\mathbf{q}\,, g)$, then not only do corresponding periodic points from the two collections have equal periods, but also the way in which they are embedded in the dynamics of the homeomorphisms relative to A and to each other are preserved. For example, in the surface case, the two collections have the same braid types and the same mutual linking numbers [16]. It is for this reason that strong Nielsen equivalence is a good relation to use in the definition of unremovability.

Definition 2.3 A periodic point collection $(\mathbf{p}\,, f) \in$ PPC is *unremovable* if for all $g \in \mathrm{Aut}(M, A)$ isotopic to f (rel A), there is some $\mathbf{q} \in P(g)$ such that $(\mathbf{p}\,, f) \sim (\mathbf{q}\,, g)$.

Remarks 2.2

a) Notice that the definition of $P(f)$ requires that each p_α lies on a different orbit. Thus, for example, if $(\mathbf{p}\,, f)$ consists of two periodic points p_1 and p_2 with $(p_1\,, f) \sim (p_2\,, f)$, then it is possible that $(\mathbf{p}\,, f)$ is not unremovable even though $(p_1\,, f)$ and $(p_2\,, f)$ are.

b) If $(\mathbf{p}, f)$ is unremovable, then so is each $(\mathbf{q}, g)$ with $(\mathbf{q}, g) \sim (\mathbf{p}, f)$.

Definitions 2.4 Let $(\mathbf{p}, f) \in \mathrm{PPC}$ and $(\mathbf{q}, g) \in \mathrm{FIX}$. A *strong approximating family from* $(\mathbf{p}, f)$ *to* $(\mathbf{q}, g)$ is a triple $(f_t, t_i, \mathbf{q}^i)$, where $f_t: f \simeq g$ is an isotopy relative to A; $t_i \to 1$ is a sequence in $[0,1]$; and $\mathbf{q}^i \to \mathbf{q}$ is a sequence in M^k with $\mathbf{q}^i \in P(f_{t_i})$ and $(\mathbf{p}, f) \sim (\mathbf{q}^i, f_{t_i})$ for all i. An *approximating family from* $(\mathbf{p}, f)$ *to* $(\mathbf{q}, g)$ is a similar triple in which f_t need only be a homotopy, $\mathbf{q}^i$ need only lie in $\mathrm{Fix}(\mathcal{F}(f_{t_i}))$, and each $(\mathbf{q}^i, f_{t_i})$ need only be Nielsen equivalent to $(\mathbf{p}, f)$.

The periodic point collection $(\mathbf{p}, f)$ is said to be *uncollapsible (relative to A)* if whenever $(\mathbf{q}, g) \in \mathrm{FIX}$ and there is a strong approximating family from $(\mathbf{p}, f)$ to $(\mathbf{q}, g)$, then $(\mathbf{q}, g) \in \mathrm{PPC}$. It is said to be *strong Nielsen bounded away from A* (respectively *Nielsen bounded away from A*) if whenever $(\mathbf{q}, g) \in \mathrm{FIX}$ and there is a strong approximating family (respectively an approximating family) from $(\mathbf{p}, f)$ to $(\mathbf{q}, g)$, then $\mathbf{q} \in A^c$.

Remarks 2.3

a) Since the conditions for an approximating family are weaker than those for a strong approximating family, it follows that Nielsen boundedness implies strong Nielsen boundedness.

b) Conditions which imply uncollapsibility, Nielsen boundedness and strong Nielsen boundedness are given in section 3.

c) If $(\mathbf{p}, f)$ is uncollapsible or strong Nielsen bounded, then so is any periodic point collection $(\mathbf{q}, g)$ which is strong Nielsen equivalent to $(\mathbf{p}, f)$.

Recall (see for example [23]) that, generalizing the notion of the index of an isolated fixed point, there is an *index function* which assigns an integer $\mathrm{index}(M, U, f)$ to each triple (M, U, f) consisting of a compact manifold M, an open subset $U \subseteq M$, and a map $f: M \to M$ with $\mathrm{Fix}(f) \cap \partial U = \emptyset$. The index is defined axiomatically, the properties which are relevant here being the following:

Normalization $\mathrm{index}(M, M, f) = L(f)$, the Lefschetz number of f.

Existence If $\mathrm{index}(M, U, f) \neq 0$, then $\mathrm{Fix}(f) \cap U \neq \emptyset$.

Homotopy If $f_t: f_0 \simeq f_1$ is a homotopy such that $\mathrm{Fix}(f_t) \cap \partial U = \emptyset$ for all t, then

$$\mathrm{index}(M, U, f_0) = \mathrm{index}(M, U, f_1).$$

Excision If V is an open subset of U such that $\mathrm{Fix}(f) \cap U = \mathrm{Fix}(f) \cap V$, then

$$\mathrm{index}(M, U, f) = \mathrm{index}(M, V, f).$$

Product $\mathrm{index}(M \times N, U \times V, f \times g) = \mathrm{index}(M, U, f) \cdot \mathrm{index}(N, V, g)$.

If F is an open subset of $\mathrm{Fix}(f)$ which is compact (for example an isolated fixed point), then its index can be defined by $\mathrm{index}(F, f) = \mathrm{index}(U, f)$, where U is an open subset of X with $U \cap \mathrm{Fix}(f) = F$. This definition is independent of the choice of U by the excision property.

Definitions 2.5 If $\mathrm{snc}(\mathbf{p}, f)$ is open and closed in $\mathrm{Fix}(\mathcal{F}(f))$, then the *index* of $(\mathbf{p}, f)$ is $I(\mathbf{p}, f) = \mathrm{index}(\mathrm{snc}(\mathbf{p}, f), \mathcal{F}(f))$. If $I(\mathbf{p}, f)$ exists and is non-zero, then $(\mathbf{p}, f)$ is said to be *essential*.

As a consequence of Proposition 2.3 below, the index always exists when $(\mathbf{p}, f)$ is uncollapsible and strong Nielsen bounded. The unremovability theorem (Theorem 2.4) states that if $(\mathbf{p}, f)$ is uncollapsible, strong Nielsen bounded, and essential, then it is unremovable.

2.2 Preliminaries to the unremovability theorem

In this section some preliminary propositions will be stated and proved. The following notation is adopted: given $\epsilon > 0$ and a subset S of M, the open and closed ϵ-neighborhoods of S are denoted $N(S, \epsilon)$ and $\overline{N}(S, \epsilon)$ respectively. An isotopy f_t is said to be an *ϵ-isotopy* if $\rho(f_s, f_t) < \epsilon$ for all s and t: if such an ϵ-isotopy exists, then f_0 and f_1 are said to be *ϵ-isotopic*. Closed disks, circles and annuli centered at the origin in $\mathbb{R}^n$ are denoted $D(\eta) = \{z \in \mathbb{R}^n : |z| \leq \eta\}$, $S_\eta = \{z \in \mathbb{R}^n : |z| = \eta\}$, and $A(a, b) = \{z \in \mathbb{R}^n : a \leq |z| \leq b\}$. If f is a map defined in a neighborhood of S_η, then f_η denotes its restriction to S_η.

The following topological lemma will be used in proving Proposition 2.2. Its proof uses a deep topological result, the annulus conjecture (now a theorem, see the comments in [25] inserted in the version reprinted in the appendix of [26] as well as [29]). We have a more elementary proof of Proposition 2.2 which is not based on the annulus conjecture, but it is much longer.

Lemma 2.1 *Let $f : D(1) \to \mathbb{R}^n$ be an orientation preserving topological embedding with $f(0) = 0$, and let $\eta \in (0, 1)$. Then there exist $\epsilon \in (0, \eta)$ and $\widehat{f} : D(1) \to \mathbb{R}^n$ with $\widehat{f} \simeq f$* rel $\{0\}$ *by an isotopy supported on $\bar{N}(0, \eta)$, and $\widehat{f} = id$ on $N(0, \epsilon)$.*

Proof. Pick $\epsilon > 0$ small enough that $D(\epsilon)$ is contained in $f(N(0, \eta))$. Now f_η is certainly a locally flat embedding since it is the restriction of a embedding of $D(1)$, so the annulus conjecture yields a homeomorphism $h : \hat{A} \to A(\epsilon, \eta)$, where $\hat{A}$ is the closed region bounded by S_ϵ and $f(S_\eta)$.

The construction of $\widehat{f}$ uses a homeomorphism $F : A(\epsilon, \eta) \to A(\epsilon, \eta)$ which sends each circle S_β to itself. Since $h f_\eta$ is an orientation preserving homeomorphism of S_η there is an

Alexander isotopy $H_t : hf_\eta \simeq id$. Pick $\alpha < (\eta - \epsilon)/2$ and for each $\eta - \alpha \le \beta \le \eta$, define $F_\beta = H_{\phi(\beta)}$ with $\phi : [\eta - \alpha, \eta] \to [0,1]$ an orientation *reversing* homeomorphism. Similarly, using an isotopy of h_ϵ to the identity, F_β can be defined for $\epsilon \le \beta \le \epsilon + \alpha$ so that $F_\epsilon = h_\epsilon$ and $F_{\epsilon+\alpha}$ is the identity. Finally, for $\epsilon + \alpha \le \beta \le \eta - \alpha$, let $F_\beta = \mathrm{id}$.

The required $\widehat{f}$ is defined by letting it equal f on $A(\eta, 1)$, $h^{-1}F$ on $A(\epsilon, \eta)$ and the identity on $D(\epsilon)$. Since $(\widehat{f}^{-1}f)_\eta = \mathrm{id}$ and $\widehat{f}^{-1}f(0) = 0$ the Alexander trick yields $\widehat{f}^{-1}f \simeq \mathrm{id} \mathrm{rel}\ \{0\}$ and so $f \simeq \widehat{f} \mathrm{rel}\ \{0\}$. ∎

Note that if H^n is the upper half space $\{x \in \mathbb{R}^n : x_n \ge 0\}$ then the analog of the proposition for embeddings $f : D(1) \cap H^n \to H^n$ also holds.

Two main propositions are prerequisites for the proof of the unremovability theorem. The first is a local result, saying that if $(\mathbf{p}\,, f)$ is a periodic point collection with $\mathbf{p} \in A^c$, then all nearby periodic point collections are strong Nielsen equivalent to $(\mathbf{p}\,, f)$.

Proposition 2.2 *Suppose that* $(\mathbf{p}\,, f) \in \mathrm{PPC}$ *with* $\mathbf{p} \in A^c$. *Then there exists* $\epsilon > 0$ *such that whenever* g *is* ϵ*-isotopic to* f rel A, *and* $\mathbf{q} \in P(g)$ *is* ϵ*-close to* $\mathbf{p}$, *then* $(\mathbf{p}\,, f) \sim (\mathbf{q}\,, g)$. *In particular, if* $(\mathbf{p}\,, f)$ *is uncollapsible and strong Nielsen bounded, and there exists a strong approximating family from* $(\mathbf{p}\,, f)$ *to* $(\mathbf{q}\,, g)$, *then* $(\mathbf{p}\,, f) \sim (\mathbf{q}\,, g)$.

Proof. Assume first that the periodic point collections $(\mathbf{p}\,, f)$ and $(\mathbf{q}\,, g)$ consist of single fixed points p and q and that p is in the interior of M. Pick $\delta > 0$ so that $N(p, \delta)$ is bounded away from A and is a coordinate chart in the manifold M. Now pick $\eta < \delta/5$ so that $f(N(p,\eta)) \subset N(p, \delta)$. Using Lemma 2.1 locally, obtain the corresponding ϵ and a local homeomorphism $\widehat{f}$ that agrees with f outside $N(p,\eta)$. Defining $\widehat{f}$ to be f outside $N(p,\eta)$ yields a homeomorphism $\widehat{f} \simeq f$ rel $(\{p\} \cup A)$ by an isotopy supported on $N(p,\eta)$ and $\widehat{f} = id$ on $N(p, \epsilon)$.

Thus $(p, f) \sim (p, \widehat{f})$ by the constant path and an isotopy supported on $N(p,\eta)$. If $f_t : f \simeq g$ is an ϵ-isotopy and (q, g) is a fixed point with $d(q,p) < \epsilon$, then $(q, \widehat{f}) \sim (p, \widehat{f})$ by any path contained in $N(0, \epsilon)$ and the constant isotopy. It remains to show that $(q, \widehat{f}) \sim (q, g)$.

If f'_t denotes the isotopy from $\widehat{f}$ to f followed by that from f to g, then $f'_t : \widehat{f} \simeq g$ is a $(\eta + \epsilon)$-isotopy. Thus, in particular, the path $t \mapsto f'_t(q)$ is contained in $N(q, 2\eta)$. Pick local coordinates (and a vector space structure) adapted to the metric for $N(q, 4\eta) \subset N(p, 5\eta) \subset N(p, \delta)$ with origin $q = 0$. For z near q define

$$r(z) = \begin{cases} 1 - |z|/2\eta & \text{if } |z| \le 2\eta \\ 0 & \text{otherwise,} \end{cases}$$

and let $k_t(z) = z - r(z - f_t'(q))f_t'(q)$, where the subtraction and multiplication use the local vector space structure. It is easy to check that k_t is an isotopy id $\simeq$ id supported on $\bar{N}(q, 4\eta)$, with $k_t(f_t'(q)) = q$ for all t. The isotopy $k_t \circ f_t' : \widehat{f} \simeq g$ is thus rel $\{q\}$, and so $(q, \widehat{f}) \sim (q, g)$ using this isotopy and the constant path.

If p is on the boundary, find $\widehat{f}$ and ϵ using the remark after Lemma 2.1. If q is in the interior, proceed as above. If q is on the boundary, then the path $t \mapsto f_t'(q)$ is also, and the isotopy k_t can be constructed as above. If p is a periodic point instead of a fixed point, choose the initial δ so that the balls $N(f^i(p), \delta)$ are disjoint charts. The generalization to periodic point collections is straightforward.

To prove the second sentence of the proposition, note that $(\mathbf{q}\,, g) \in \mathrm{PPC}$ and $\mathbf{q} \in A^c$ by uncollapsibility and strong Nielsen boundedness respectively. By the first part of the proposition, $(\mathbf{q}\,, g)$ is strong Nielsen equivalent to all sufficiently close periodic point collections in the strong approximating family, and hence $(\mathbf{p}\,, f) \sim (\mathbf{q}\,, g)$ by the transitivity of $\sim$. ∎

The second proposition is a standard form which is needed for any theory of Nielsen type. It says that the strong Nielsen class of an uncollapsible and strong Nielsen bounded periodic point collection $(\mathbf{p}\,, f)$ is both open and closed in the appropriate fixed point set, and thus its index is defined. The index is also defined for any $(\mathbf{q}\,, g) \sim (\mathbf{p}\,, f)$, by Remark 2.3 c): the second statement of the proposition says that the index is independent of the choice of periodic point collection in the strong Nielsen equivalence class. The following notation is used in the proof of this proposition: if $\bowtie$ is a binary relation, and $a(i)$ and $b(i)$ are expressions dependent on a positive integer i, then $a(i) \bowtie_i b(i)$ means that $a(i) \bowtie b(i)$ for all sufficiently large i.

Proposition 2.3 *If* $(\mathbf{p}\,, f)$ *is uncollapsible and strong Nielsen bounded, then* $\mathrm{snc}(\mathbf{p}\,, f)$ *is compact, and open in* $\mathrm{Fix}(\mathcal{F}(f))$. *If* $(\mathbf{q}\,, g) \sim (\mathbf{p}\,, f)$ *then* $I(\mathbf{q}\,, g) = I(\mathbf{p}\,, f)$.

Proof. **Compactness:** Let $\mathbf{q}^i$ be a sequence in $\mathrm{snc}(\mathbf{p}\,, f)$ with $\mathbf{q}^i \to \mathbf{q} \in M^k$. Then there is a strong approximating family $(f_t = f, 1 - 1/i, \mathbf{q}^i)$ from $(\mathbf{p}\,, f)$ to $(\mathbf{q}\,, f)$, and hence $(\mathbf{p}\,, f) \sim (\mathbf{q}\,, f)$ by Proposition 2.2: that is, $(\mathbf{q}\,, f) \in \mathrm{snc}(\mathbf{p}\,, f)$. Thus $\mathrm{snc}(\mathbf{p}\,, f)$ is closed in the compact space M^k.

Openness in $\mathrm{Fix}(\mathcal{F}(f))$**:** Let $\mathbf{q} \in \mathrm{snc}(\mathbf{p}\,, f)$, and let $\mathbf{q}^i \in \mathrm{Fix}(\mathcal{F}(f))$ with $\mathbf{q}^i \to \mathbf{q}$. Then $\mathbf{q}^i \in_i P(f)$ by continuity, and hence $(\mathbf{q}^i\,, f) \sim_i (\mathbf{q}\,, f) \sim (\mathbf{p}\,, f)$ by Proposition 2.2. Thus $\mathbf{q}^i \in_i \mathrm{snc}(\mathbf{p}\,, f)$.

Independence of index: Suppose that $(\mathbf{p}\,, f) \sim (\mathbf{q}\,, g)$ by a path $\mathbf{p}^t$ and an isotopy f_t. For each integer m, define $I_m = \{t \in [0,1] : I(\mathbf{p}^t\,, f_t) = m\}$. The result will be proved by showing that each I_m is open in $[0,1]$.

Suppose then that there is some I_m which is not open, and pick $a \in I_m \setminus \mathrm{Int}(I_m)$. By the first part of the proof, there exist $\theta > 0$ and an open subset U of M^k with $U \cap \mathrm{Fix}(\mathcal{F}(f_a)) = \mathrm{snc}(\mathbf{p}^a, f_a)$ and $N(\mathrm{snc}(\mathbf{p}^a, f_a), \theta) \subseteq U$.

For each $i \geq 1$ define $V_i = N(\mathrm{snc}(\mathbf{p}^a, f_a), \theta/2^i)$. Then $\overline{V}_1 \setminus V_i$ is compact and disjoint from $\mathrm{Fix}(\mathcal{F}(f_a))$ for each i, so there exist numbers $\delta_i > 0$ with $d(\mathbf{x}, \mathcal{F}(f_a)(\mathbf{x})) > \delta_i$ for all $x \in \overline{V}_1 \setminus V_i$. It is therefore possible to pick a sequence $t_i \to a$ in $[0,1]$ with $t_i \notin I_m$ and $\mathrm{Fix}(\mathcal{F}(f_t)) \cap \overline{V}_1 = \mathrm{Fix}(\mathcal{F}(f_t)) \cap V_i$ for all t between t_i and a. Write f_i for f_{t_i} and $\mathbf{p}^i$ for $\mathbf{p}^{t_i}$.

Now

$$I(\mathbf{p}^a, f_a) = \mathrm{index}(V_1, \mathcal{F}(f_a)) = \mathrm{index}(V_1, \mathcal{F}(f_i)) = \mathrm{index}(V_i, \mathcal{F}(f_i))$$

for all i by the homotopy and excision properties of the index. The required contradiction will be obtained by showing that $\mathrm{Fix}(\mathcal{F}(f_i)) \cap V_i =_i \mathrm{snc}(\mathbf{p}^i, f_i)$, so that $I(\mathbf{p}^i, f_i) =_i m$.

a) To show that $\mathrm{snc}(\mathbf{p}^i, f_i) \subseteq_i \mathrm{Fix}(\mathcal{F}(f_i)) \cap V_i$: if not, then (taking a subsequence if necessary) there exist points $\mathbf{q}^i \in \mathrm{snc}(\mathbf{p}^i, f_i) \cap (M^k \setminus V_1)$ with $\mathbf{q}^i \to \mathbf{q} \in M^k \setminus V_1$. Now $(\mathbf{q}^i, f_i) \sim (\mathbf{p}^i, f_i) \sim (\mathbf{p}, f)$, and so $(f_t, t_i, \mathbf{q}^i)$ is a strong approximating family from $(\mathbf{p}, f)$ to $(\mathbf{q}, f_a)$. It follows by Proposition 2.2 that $(\mathbf{q}, f_a) \sim (\mathbf{p}, f) \sim (\mathbf{p}^a, f_a)$. Thus $\mathbf{q} \in \mathrm{snc}(\mathbf{p}^a, f_a)$, contradicting $\mathbf{q} \notin V_1$.

b) To show that $\mathrm{Fix}(\mathcal{F}(f_i)) \cap V_i \subseteq_i \mathrm{snc}(\mathbf{p}^i, f_i)$: if not, then (taking a subsequence if necessary) there exist points $\mathbf{q}^i \in \mathrm{Fix}(\mathcal{F}(f_i)) \cap V_i \setminus \mathrm{snc}(\mathbf{p}^i, f_i)$ with $\mathbf{q}^i \to \mathbf{q} \in M^k$, and $\mathbf{q} \in \mathrm{snc}(\mathbf{p}^a, f_a)$ by definition of the V_i. Thus $\mathbf{q}^i \in_i P(f_i)$, and it follows that $(\mathbf{q}^i, f_i) \sim_i (\mathbf{q}, f_a)$ by Proposition 2.2. Thus $(\mathbf{q}^i, f_i) \sim_i (\mathbf{p}^a, f_a) \sim (\mathbf{p}^i, f_i)$, so that $\mathbf{q}^i \in_i \mathrm{snc}(\mathbf{p}^i, f_i)$, which is a contradiction. ∎

2.3 The unremovability theorem

Theorem 2.4 *Let M be a compact smooth orientable manifold, and A a compact subset of M. Let $(\mathbf{p}, f)$ be a periodic point collection which is uncollapsible, strong Nielsen bounded, and essential. Then $(\mathbf{p}, f)$ is unremovable.*

Proof. Let $g \in \mathrm{Aut}(M, A)$ be isotopic to f, and pick a rel A isotopy $f_t \colon f \simeq g$. Define

$$T = \{t \in [0,1] : \text{ there exists } \mathbf{q}^t \in P(f_t) \text{ with } (\mathbf{q}^t, f_t) \sim (\mathbf{p}, f)\}.$$

The set T is closed in $[0,1]$ by Proposition 2.2: since $0 \in T$, it therefore suffices to show that T is also open in $[0,1]$.

Fix $t_0 \in T$, and let $S = \mathrm{snc}(\mathbf{q}^{t_0}, f_{t_0})$. By Proposition 2.3, S is compact and open in $\mathrm{Fix}(\mathcal{F}(f_{t_0}))$, and $\mathrm{index}(S, \mathcal{F}(f_{t_0})) \neq 0$. Choose $\theta > 0$ such that $N(S, \theta) \cap \mathrm{Fix}(\mathcal{F}(f_{t_0})) = S$.

By Proposition 2.2 and the compactness of S, there exists $\eta < \theta$ such that if $|t - t_0| < \eta$ and $\mathbf{q}^t \in \mathrm{Fix}(\mathcal{F}(f_t))$ with $d(\mathbf{q}^t, S) < \eta$, then $\mathbf{q}^t \in P(f_t)$ and $(\mathbf{q}^t, f_t) \sim (\mathbf{q}^{t_0}, f_{t_0}) \sim (\mathbf{p}, f)$.

Let $N = N(S, \eta)$. Then ∂N is compact and disjoint from $\mathrm{Fix}(\mathcal{F}(f_{t_0}))$, and hence there exists $\epsilon < \eta$ such that $\mathrm{Fix}(\mathcal{F}(f_t)) \cap \partial N = \emptyset$ whenever $|t - t_0| < \epsilon$. By the homotopy axiom of the index, it follows that for any such t there exists $\mathbf{q}^t \in N \cap \mathrm{Fix}(\mathcal{F}(f_t))$. By the previous paragraph, $\mathbf{q}^t \in P(f_t)$ and $(\mathbf{q}^t, f_t) \sim (\mathbf{p}, f)$, and hence $t \in T$ whenever $|t - t_0| < \epsilon$. Thus T is open in $[0, 1]$ as required. ∎

Remarks 2.4

a) Suppose that (p, f) is a periodic point collection consisting of a single periodic point, and that $(p, f) \sim (a, f)$ for some $a \in A$. Then (p, f) is certainly unremovable (since $(a, f) \sim (a, g)$ by the constant path whenever g is isotopic to f rel A), but the hypotheses of the theorem cannot be satisfied, as (p, f) is not strong Nielsen bounded. This case often needs to be considered separately in applications.

b) An elementary but useful observation is that if A' is a compact invariant subset of A, then a periodic point collection which is unremovable rel A' is also unremovable rel A.

3 Criteria for uncollapsibility and Nielsen boundedness

In this section a number of conditions are given which imply the hypotheses of strong Nielsen boundedness and uncollapsibility. These conditions require a variety of additional Nielsen-type definitions: Nielsen bounded away from a point in A and a regular point in A (Definition 3.1); Nielsen separated (Definition 3.2); and irreducible (Remarks 3.2 b)). Recall (Remark 2.3 a)) that, contrary to what is suggested by the terminology, Nielsen bounded implies strong Nielsen bounded.

3.1 Conditions for Nielsen boundedness

Two conditions are given which imply that a given periodic point collection is Nielsen bounded away from A. The first defines what it means for a periodic point (p, f) to be Nielsen bounded away from a specific point $a \in A$. Informally it says that not only is p not Nielsen equivalent to a, but it is never Nielsen equivalent to a periodic point (q, g) with q close to a. Proposition 3.1 b) states that being Nielsen bounded away from all compatible periodic points in A implies that a periodic point is Nielsen bounded away from A.

The second condition is not on the periodic point p, but rather is a condition on a point $a \in A$. Such a point is called *regular* if it is Nielsen equivalent to all nearby periodic points of all nearby maps. The simplest example of a regular point is an isolated point of

A. Proposition 3.1 a) states that if a periodic point is not Nielsen equivalent to a regular point $a \in A$, then it is Nielsen bounded away from it.

If $f \in \mathrm{Aut}(M, A)$, then let $[f]$ denote its rel A isotopy class.

Definition 3.1 A period n point (p, f) is said to be *Nielsen bounded away from $a \in A$*, if there exists a continuous positive function $\delta : [f] \to \mathbb{R}^+$ such that (p, f) is not Nielsen equivalent in $\mathrm{FIX}(n)$ to any periodic point (q, g) with $g \in [f]$ and $q \in N(a, \epsilon(g))$. A period n point (a, f) with $a \in A$ is called *regular* if there exists a continuous positive function $\epsilon : [f] \times \mathbb{Z} \to \mathbb{R}^+$ such that for all $m \in \mathbb{Z}$, $(q, g) \in \mathrm{FIX}(mn)$ and $d(q, a) < \epsilon(g, m)$ implies that (q, g) is Nielsen equivalent to (a, f) in $\mathrm{FIX}(mn)$.

Proposition 3.1

a) If (p, f) is a period n point which is not Nielsen equivalent to a regular point (a, f), then it is Nielsen bounded away from it.

b) If every periodic point p_α in the periodic point collection $(\mathbf{p}, f)$ is Nielsen bounded away from every periodic point $a \in A$ whose period divides that of p_α, then $(\mathbf{p}, f)$ is Nielsen bounded away from A.

c) If every periodic point p_α in an uncollapsible periodic point collection $(\mathbf{p}, f)$ is Nielsen bounded away from every periodic point $a \in A$ which is strong Nielsen equivalent to p_α relative to the empty set*, then $(\mathbf{p}, f)$ is strong Nielsen bounded away from A.*

Proof. For part a), first note that if (a, f) is not in $\mathrm{FIX}(n)$ then the result follows directly, so assume $(a, f) \in \mathrm{FIX}(n)$. It will be shown that (p, f) is Nielsen bounded away from a with a function $\delta = \epsilon(\cdot, n)$. If this is not the case then (p, f) is Nielsen equivalent to some $(q, g) \in \mathrm{FIX}(n)$ with $d(q, a) < \epsilon(g, n)$. By regularity, (q, g) is Nielsen equivalent to (a, f) in $\mathrm{FIX}(n)$. By transitivity, (p, f) is Nielsen equivalent to (a, f), a contradiction.

For part b), assume first that the periodic point collection consists of a single period n point p which is Nielsen bounded away from every periodic point in A whose period divides n, but is not Nielsen bounded away from A. Then there exists an approximating family $(f_t \colon f \simeq g, t_i, q^i)$ with $q^i \to a \in A$. By continuity, a is a periodic point of g, and hence of f, with period dividing n, and so (p, f) is Nielsen bounded away from it. However, $\epsilon(f_{t_i}) \to \epsilon(g) > 0$ and $d(q^i, a) \to 0$, so (p, f) cannot be Nielsen equivalent to (q^i, f_{t_i}) for large i, a contradiction. The generalization to periodic point collections with multiple members is immediate.

The proof of part c) is similar to that of part b). Assume that the periodic point collection consists of a single period n point p which is Nielsen bounded away from every periodic point in A which is strong Nielsen equivalent to p relative to the empty set. If

(p, f) is not strong Nielsen bounded away from A, then there is a strong approximating family $(f_t: f \simeq g, t_i, q^i)$ with $q^i \to q \in A$. Since (p, f) is uncollapsible, a is a period n point of g, and hence of f: moreover $(q^i, f_{t_i}) \sim (a, g) \sim (a, f)$ relative to the empty set for i sufficiently large by proposition 2.2. Hence (p, f) is Nielsen bounded away from (a, f), and a contradiction follows as in part b). ∎

Remarks 3.1

a) If A contains no periodic points, then it follows that any periodic point collection is Nielsen bounded away from A.

b) If there exists $B \subseteq A$ such that $N(B, \epsilon) \cap A$ is f-invariant for arbitrarily small ϵ, then it may be possible to prove unremovability rel $A \setminus N(B, \epsilon)$ as in Remark 2.4 b).

c) By Proposition 3.1 c), if M is a surface and $(\mathbf{p}, f)$ is an uncollapsible periodic point collection, then it is only necessary to check that each point in the collection is Nielsen bounded away from all of the periodic points of A which have the same braid type (see [6] for the definition of braid type and related information).

3.2 Criteria for uncollapsibility

In this section a criterion for uncollapsibility is given which is frequently useful in applications. It is assumed that the periodic point collection is strong Nielsen bounded away from A, so that the argument focuses on behavior in the complement. The subsequent remarks (Remarks 3.2) contain brief descriptions of two other methods of showing uncollapsibility.

Definition 3.2 Let $(\mathbf{p}, f) \in \mathrm{PPC}$ with $\mathbf{p} \in A^c$. Then $(\mathbf{p}, f)$ is *Nielsen separated* rel A if any two distinct points of $\bigcup_\alpha o(p_\alpha, f)$ which have the same period n lie in distinct f^n-Nielsen classes rel A.

Lemma 3.2 *Let $f \in \mathrm{Aut}(M, A)$, and p_0 and p_1 be fixed points of f^n lying in A^c. Then there exists $\epsilon > 0$ such that if $g \in \mathrm{Aut}(M, A)$ is ϵ-isotopic to f* rel A, *and q_0 and q_1 are fixed points of g^n with $d(p_i, q_i) < \epsilon$ for each i, then p_0 and p_1 are in the same f^n-Nielsen class* rel A *if and only if q_0 and q_1 are in the same g^n-Nielsen class* rel A.

Proof. Suppose first that p_0 and p_1 lie in the same f^n-Nielsen class, and let γ be a path in A^c from p_0 to p_1 with $f^n(\gamma) \simeq \gamma$ rel A. Since γ and $f^n(\gamma)$ are bounded away from A, it follows that if ϵ is sufficiently small then the path η from q_0 to q_1 obtained from γ by adjoining short paths from q_0 to p_0 and from p_1 to q_1 satisfies $g^n(\eta) \simeq \eta$.

For the converse, suppose that p_0 and p_1 lie in distinct f^n-Nielsen classes. Let N_0 and N_1 be disjoint coordinate neighborhoods of these two points, each disjoint from A. If ϵ is small enough then there are paths λ_i in N_i from p_i to q_i, whose images under g^n also lie in N_i. Now suppose that q_0 and q_1 lie in the same g^n-Nielsen class, and let γ be a path from q_0 to q_1 with $g^n(\gamma) \simeq \gamma$. Write $\eta = \lambda_0 \cdot \gamma \cdot \lambda_1^{-1}$. Since $f^n(\gamma)$ is homotopic to $g^n(\gamma)$ by a homotopy in which the endpoints remain in $N_0 \cup N_1$, it follows that $f^n(\eta) \simeq \eta$, a contradiction. ∎

Lemma 3.3 *Suppose that* $(\mathbf{p}, f)$ *is strong Nielsen bounded. If* $(\mathbf{p}, f)$ *is Nielsen separated, then so is* $(\mathbf{q}, g)$ *whenever* $(\mathbf{p}, f) \sim (\mathbf{q}, g)$.

Proof. Let α and β be such that $n_\alpha = n_\beta = n$, and i and j be integers between 0 and $n-1$. Suppose that $(\alpha, i) \neq (\beta, j)$. It is required to show that $g^i(q_\alpha)$ and $g^j(q_\beta)$ lie in distinct f^n-Nielsen classes rel A.

Let $(\mathbf{p}, f) \sim (\mathbf{q}, g)$ by a path $\mathbf{p}^t$ and an isotopy $f_t\colon f \simeq g$. Let S be the set of all $t \in [0,1]$ such that $f_t^i(p_\alpha^t)$ and $f_t^j(p_\beta^t)$ lie in distinct f_t^n-Nielsen classes rel A. Then $0 \in S$, and S is open and closed in $[0,1]$ by Lemma 3.2. ∎

Proposition 3.4 *If* $(\mathbf{p}, f)$ *is strong Nielsen bounded and Nielsen separated, then it is uncollapsible.*

Proof. Suppose to the contrary that $(\mathbf{p}, f)$ is strong Nielsen bounded and Nielsen separated, but is not uncollapsible. Let $(f_t, t_i, \mathbf{q}^i)$ be a strong approximating family from $(\mathbf{p}, f)$ to $(\mathbf{q}, g)$, where $\mathbf{q} \notin P(g)$. Since $(\mathbf{p}, f)$ is strong Nielsen bounded, it follows that $\mathbf{q} \in A^c$.

Suppose first that there is some α with $q_\alpha \notin P_{n_\alpha}(g)$. Then for i sufficiently large, there are distinct points of the orbit of q_α^i which lie in the same $f_{t_i}^{n_\alpha}$-Nielsen class by Lemma 3.2 applied to g (taking $p_0 = p_1 = q_\alpha$). This contradicts Lemma 3.3.

If this is not the case, then there exist distinct indices α and β such that the orbits of q_α and q_β coincide (and both have period $n_\alpha = n_\beta$, since the first case does not occur). Using Lemma 3.2 again, this implies that for i sufficiently large there are points on the orbits of q_α^i and q_β^i which lie in the same $f_{t_i}^{n_\alpha}$-Nielsen class, contradicting Lemma 3.3. ∎

Remarks 3.2

a) Fixed points are trivially uncollapsible.

b) A periodic point collection $(p, f) \in \mathrm{PPC}(n)$ consisting of a single period n point is said to be *irreducible* if there is no proper divisor m of n such that there exists a path γ from p to $f^m(p)$ for which the loop $\gamma \cdot f^m(\gamma) \cdot \ldots \cdot f^{n-m}(\gamma)$ is homotopically trivial (see for example [23]). If $(\mathbf{p}, f)$ has the property that the periods of all the points p_α

are distinct, then say that $(\mathbf{p}, f)$ is irreducible if each (p_α, f) is irreducible. Suppose all of the periods in $(\mathbf{p}, f)$ are distinct, and that $(\mathbf{p}, f)$ is strong Nielsen bounded. Then it can be shown that if $(\mathbf{p}, f)$ is Nielsen separated, then it is irreducible; and if it is irreducible, then it is uncollapsible. There does not seem to be a sensible extension of the definition of irreducibility to the case in which the periods of the points in the collection are not all distinct.

c) A common situation in examples is that for each $n \geq 1$, all of the fixed points of f^n are in different Nielsen classes. This implies that any periodic point collection is Nielsen separated. Hence for such a map any periodic point collection which is strong Nielsen bounded is uncollapsible.

d) It is often the case in two-dimensional applications that $(\mathbf{p}, f)$ can be shown to be uncollapsible by using the mutual linking properties of the different periodic orbits in the collection. Suppose, for example, that $M = D^2$, that p_1 and p_2 are fixed points of f which lie in different f-Nielsen classes rel A, and that p_3 is a period n orbit of f which has rotation numbers m_1/n and m_2/n in the annuli $D^2 \setminus \{p_1\}$ and $D^2 \setminus \{p_2\}$, with $(m_1, n) = (m_2, n) = 1$. Then if $((p_1, p_2, p_3), f)$ is strong Nielsen bounded, it is uncollapsible. For the continuations of p_1 and p_2 cannot collide, since they lie in different Nielsen classes; the continuation of p_3 cannot period-divide except onto the continuation of p_1, since otherwise its rotation number about p_1 would have to change discontinuously; and similarly the continuation of p_3 cannot period-divide except onto the continuation of p_2. Similar arguments can sometimes be applied inductively to a large collection of periodic points, using the linking of the orbit of each p_α about that of $p_{\alpha-1}$ to rule out period-division of the continuation of p_α (see for example the proof of lemma 9.9 of [17]).

4 Examples and Applications

4.1 First Examples

The two examples in this section are intended to illustrate some of the definitions, and to show simple applications of the main theorem and propositions. In the first, Theorem 2.4 is used to show the unremovability of a pair of fixed points of a homeomorphism relative to an invariant set which is minimal (i.e. every orbit in A is dense in A). In the second example, A has a limit point which is fixed, and a fixed point in A^c is seen to be unremovable after showing that it is unremovable relative to an invariant subset of A, illustrating Remark 3.1 b).

Both examples are homeomorphisms of the two-sphere S^2, regarded as the Riemann sphere $\mathbb{C} \cup \{\infty\}$. The origin and the point ∞ become the south and north pole, denoted S and N respectively.

For the first example, start with a homeomorphism $g: S^1 \to S^1$ which is a Denjoy counterexample, so that g has a minimal invariant Cantor set (see for example [10]). Define $G: \mathbb{C} \to \mathbb{C}$ by $G(z) = |z| \exp(ig(\arg(z)))$, and extend G to a homeomorphism of S^2. Let A be the minimal invariant Cantor set in the unit circle (the equator). It will be shown that $((N, S), G)$ is unremovable relative to A. Observe first that it is strong Nielsen bounded away from A, since A contains no periodic points (Remark 3.1 a)). Now N and S lie in distinct G-Nielsen classes rel A. To see this, note that for any path γ from N to S in A^c, the first gap of A through which it passes is a well-defined, homotopy invariant notion. The action of G is such that this gap is different for $f(\gamma)$, and so γ and $f(\gamma)$ are never homotopic rel endpoints. Thus $((N, S), G)$ is Nielsen separated, and hence uncollapsible by Proposition 3.4. Finally, each of the two fixed points has G-index $+1$, and hence $I((N, S), G) = 1$ by the product property of the index. The periodic point collection is therefore essential, and hence unremovable by Theorem 2.4. In particular, any homeomorphism which is isotopic to Grel A has at least two fixed points.

For the second example, fix for each $n \geq 1$ a circle homeomorphism $h_n: S^1 \to S^1$ such that the rotation number $\rho(h_n)$ is non-zero, and that h_n has either a periodic orbit or a Denjoy minimal set (equivalently, h_n does not have a fixed point or a dense orbit). Define H on the circle $|z| = 1/n$ by $H(z) = \exp(ih_n(\arg(z)))/n$, and extend to a homeomorphism $H: \mathbb{C} \to \mathbb{C}$ which has only two fixed points, N and S. For each $n \geq 1$, let A_n be a periodic orbit or Denjoy minimal set in the circle By a similar argument to that used in the first example, it can be seen that N is is unremovable relative to A_n for any n, and thus is unremovable rel $A = \bigcup_n A_n \cup \{S\}$.

4.2 Nielsen equivalence to points in A

This example illustrates the importance of the Nielsen boundedness hypothesis. A positive index fixed point p in A^c, which is alone in its Nielsen class and is not Nielsen equivalent to any point in A is shown to be removable. By Theorem 2.4 it follows that p is not bounded away from A: in fact, it is not Nielsen bounded away from any $a \in A$.

Let $B = [0, 2] \times S^1$ be an annulus of width 2 with coordinates (s, θ), and denote its interior $B^o = (0, 2) \times S^1$. Define $\alpha: (0, 2) \to \mathbb{R}$ by

$$\alpha(s) = \begin{cases} \log(s) & \text{if } \ 0 < s \leq 1 \\ -\log(2 - s) & \text{if } \ 1 \leq s < 2, \end{cases}$$

and $\phi : B^o \to B^o$ by $\phi(s, \theta) = (s, \theta + \alpha(s))$. Thus ϕ twists the interval $(0, 2) \times \{0\}$ around the annulus infinitely many times.

Define a class of *allowable* homeomorphisms $g : B \to B$

$$g(s, \theta) = (s + \beta(s, \theta), \theta + \gamma(s, \theta))$$

by imposing that β and γ satisfy

$$\frac{\beta(s,\theta)}{s} \to 0, \quad \gamma(s,\theta) \to 0$$

as $s \to 0$ uniformly in θ and analogous conditions near the other boundary of B: the conditions on γ just say that g is the identity on ∂B. Notice that, given any flow on B which is fixed on ∂B, the time parameterization may be adjusted so that the time one map of the flow is an allowable homeomorphism.

For any allowable g, define

$$G(z) = \begin{cases} \phi \circ g \circ \phi^{-1} & \text{if } z \in B^o \\ z & \text{if } z \in \partial B. \end{cases}$$

Then the conditions on $\beta(s,\theta)$ and $\gamma(s,\theta)$ imply that G is a homeomorphism of B. To see this, simply write out the expression for $G(s,\theta)$ explicitly in the case $s \in (0,1)$, and observe that $G(s,\theta) \to G(0,\theta)$ as $s \to 0$ and $G(s,\theta) \to G(2,\theta)$ as $s \to 2$ for each fixed θ. Hence G is a continuous bijection, and therefore a homeomorphism.

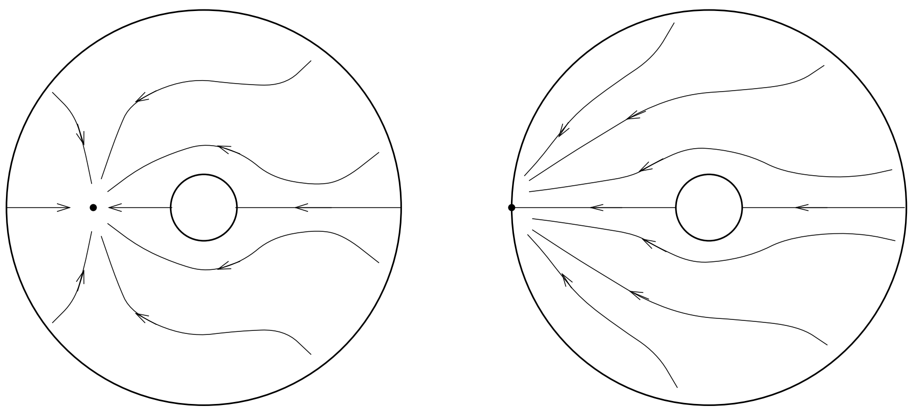

Figure 1: Flows on the annulus with time one maps g_0 (left) and g_1 (right)

Now let $\psi_{0,t}$ and $\psi_{1,t}$ be flows with trajectories as shown in Figure 1 a) and b) respectively. Adjust the time parameterization so that their time one maps, g_0 and g_1, are allowable homeomorphisms, and so that g_0 and g_1 agree on $I = [0,2] \times \{0\}$. By continuously deforming the first flow into the second, an isotopy $g_t\colon g_0 \simeq g_1$ relative to $I \cup \partial B$ can be constructed in such a way that each g_t is also allowable. The isotopy $G_t = \phi \circ g_t \circ \phi^{-1}$ on B^o can thus be extended to the identity on ∂B, and this extended isotopy is relative to $\phi(I^o) \cup \partial B$. Let $A = \phi(X) \cup \partial B$, where $X = o(x, g_0) = o(x, g_1)$ is the orbit of some point x in the interior of I.

Let p be the unique fixed point of G_0 in A^c, which is alone in its G_0-Nielsen class relative to A and has index $+1$. It must be removable rel A as G_1 has no fixed point in A^c. Further, it is easy to check that (p, G_0) is not Nielsen equivalent to any $a \in A$. Informally speaking, p has been removed by an isotopy which pushes it onto all of A despite the fact that it is not Nielsen equivalent to any point in A.

4.3 An infinite orbit with an unremovable horseshoe

This example is due to Michael Handel (personal communication and [20]), and illustrates what he calls a 'homotopy horseshoe'. It may be viewed as arising from a Nielsen-Thurston type theory for mapping classes with 'translation ends' on noncompact surfaces [21] (*cf.* [12]). The example consists of a homeomorphism $\Psi\colon D^2 \to D^2$ of the two-disk. The compact invariant set A is the closure of a single orbit traveling from one point on ∂D^2 to another. The nonwandering set Λ of Ψ in the interior of D^2 is a Smale horseshoe, and is therefore a Cantor set on which the dynamics is conjugate to the full shift on two symbols: it will be shown that all of the periodic points in Λ are unremovable relative to A.

Write $D^2 = [0,1]\times[-1,1]$, and define $T\colon D^2 \to D^2$ by $T(x,y) = (\sqrt{x}, y)$. Set $x_0 = (1/2, 0)$, and write $x_i = T^i(x_0)$ for each integer i, so that $X = \{x_i : i \in \mathbb{Z}\}$ is the orbit of x_0 under T. Let A be the closure of X, so that $A = X \cup \{\alpha\} \cup \{\omega\}$, where $\alpha = (0,0)$ and $\omega = (1,0)$.

The rel A isotopy class of Ψ will be that of $D \circ T$, where D is a clockwise Dehn twist about a simple closed curve which is contained in $(1/4, 1/\sqrt[4]{2}) \times [-1,1]$, and bounds a disk containing x_0 and x_1. The construction of an explicit map Ψ in this isotopy class generalizes that of [13] from the case of a periodic orbit to that of an infinite discrete orbit. For each $i \in \mathbb{Z}$, let $R_i = [a_i, b_i] \times [-1/2, 1/2]$ be a 'thin' rectangle which contains x_i but no other points of X, and let $B_i = [b_i, a_{i+1}] \times [-1/2, 1/2]$ be a rectangle which joins R_i to R_{i+1}. Then construct Ψ in the isotopy class of $D \circ T$ in such a way that (see figure 2)

a) $\Psi(R_i) \subseteq R_{i+1}$, and Ψ acts as a strict contraction on R_i for each i.

b) $\Psi(B_i) \cap B_j$ is only non-empty when this is required by the isotopy class: that is $\Psi(B_i)$ intersects only B_{i+1}, except in the cases $i = -2$ and $i = 0$, where $\Psi(B_i)$ also intersects B_0.

c) When $\Psi(B_i) \cap B_j \neq \emptyset$, then horizontal and vertical lines in $B_i \cap \Psi^{-1}(B_j)$ are sent to horizontal and vertical lines in B_j, with a uniform expansion and contraction respectively.

d) Ψ agrees with T on ∂D^2: all of the interior periodic points of Ψ lie in the rectangles B_i (in fact they all lie in B_0).

Notice that the image of B_0 crosses over B_0 as in a Smale horseshoe. Let Λ be the resulting compact invariant set (i.e. the set of points whose entire orbit lies in B_0). Every periodic point in Λ has index ± 1. Moreover, it can be shown (see the next paragraph), first, that all of these periodic points are Nielsen bounded away from α and ω (with the exception of the fixed point labeled q in figure 2, which is strong Nielsen equivalent to all of the fixed points on ∂D^2); and second, that for each n, all of the fixed points of Ψ^n in Λ

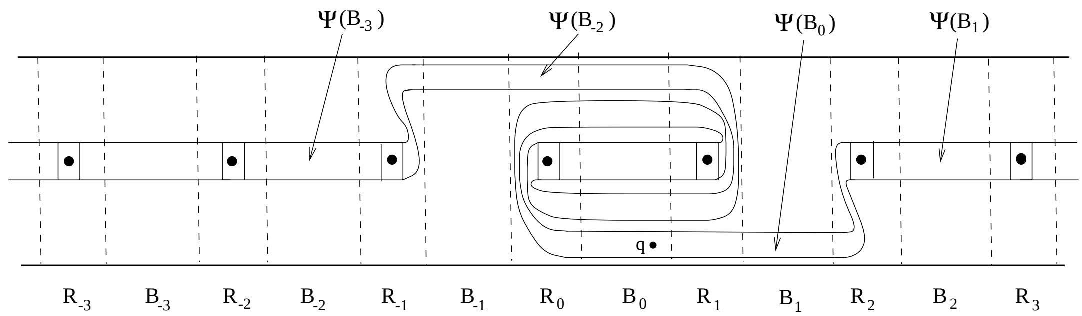

Figure 2: Construction of the homeomorphism Ψ

lie in distinct Ψ^n-Nielsen classes rel A. Since the fundamental group of $D^2 \setminus A$ is free, and hence has trivial center, this second fact also implies that each of the periodic points in Λ is alone in its strong Nielsen class (Remark 2.1 f)). Hence any periodic point collection $(\mathbf{p}, \Psi)$ for which each p_α is an interior periodic point distinct from q is strong Nielsen bounded, uncollapsible (by Proposition 3.4), and essential, and hence is unremovable by Theorem 2.4. In particular, any homeomorphism in this isotopy class has a periodic orbit of each braid type which is present in Smale's horseshoe.

There are at least two ways to prove the two assertions in the previous paragraph. The first is to use the action of Ψ on $\pi_1(D^2 \setminus A)$ in conjunction with the symbolic description of Λ to compute twisted conjugacy classes directly. Different periodic points in Λ will have nonequivalent classes implying that the periodic points are in different Nielsen classes. Further, none of these classes contain a representative containing just the generators near α or those near ω. This implies that the periodic points cannot be Nielsen equivalent to a periodic point near α or ω, i.e. the periodic points are bounded away from α and ω. The second method of proof [22] is to adapt the Bestvina-Handel algorithm [3] to the case of such an infinite orbit X to construct a pseudo-Anosov-like representative Φ of the isotopy class which is essentially the same as Ψ (see [21, 12]), and then to imitate the usual proof for pseudo-Anosov maps by lifting the invariant laminations to the universal cover [19].

4.4 Unremovability relative to a generalized adding machine

In this section, a homeomorphism $\Phi\colon D^2 \to D^2$ is defined which has a minimal invariant Cantor set A, relative to which all of the periodic points of Φ are unremovable. The restriction of Φ to A is conjugate to a generalized adding machine, but (even when a particular conjugacy class of adding machine has been chosen) there are infinitely many choices during the construction which influence the isotopy class of Φ relative to A. From the point of view of Nielsen-Thurston theory, Φ may be thought of as the analogue of a Nielsen-Thurston canonical representative of a reducible isotopy class which has infinitely many components. For the sake of brevity, it will be assumed that the reader is familiar with Nielsen-Thurston theory: more information can be found in [11, 6].

The restriction $\Phi: A \to A$ falls into the class of minimal dynamical systems which are termed generalized adding machines (the terms odometer and dial are also used). The suspension flows of these systems are called generalized solenoids. Generalized adding machines have a long history in dynamics. They arise naturally in area preserving maps of the plane, where one finds infinite nested families of elliptic periodic orbits, each one rotating about the one before it in the family. The adding machine arises as a limit of these orbits (see [4] Section 7). The suspension of this situation arises generically in Hamiltonian flows [27]. Generalized adding machines as minimal dynamical systems have also been studied extensively in the topological dynamics literature. Buescu and Stewart [8] show that whenever A is a compact Lyapunov stable transitive invariant set for a discrete dynamical system $f: X \to X$, where X is locally compact and locally connected, the dynamics of f on the components of A is topologically conjugate to a generalized adding machine.

The construction of a disk homeomorphism detailed here is a generalization of the situation on which Birkhoff commented, and is quite familiar from examples of Kupka-Smale diffeomorphisms with no sources or sinks [5, 14, 30] and related constructions [15]. The first step is to define a family of nested disks in D^2 on which the action of Φ will subsequently be defined. The nested disks will be indexed symbolically in the following way. Let $n_0 = 1$, and fix a list $(n_i)_{i\geq 1}$ of integers with $n_i > 1$ for each i. For each $k \geq 0$, define $S_k \subseteq \mathbb{Z}^k$ by $S_k = \{(s_1, s_2, \ldots, s_k) : 1 \leq s_i \leq n_i\}$, and extend this definition to the case $k = \infty$ in the obvious way. An *allowable sequence* s is defined to be an element of S_k for some $k \in \mathbb{N} \cup \{\infty\}$. For $1 \leq k < \infty$, the projection $p: S_k \to S_{k-1}$ is defined by dropping the last element of the sequence, so $p(s_1, \ldots, s_k) = (s_1, \ldots, s_{k-1})$.

For each $k \in \mathbb{N} \cup \{\infty\}$, the *addition map* $\alpha_k: S_k \to S_k$ adds one to s_1 and carries to the right: that is, $\alpha_k((s_1, \ldots, s_k)) = (t_1, \ldots, t_k)$, where

$$t_i = \begin{cases} s_i + 1 & \text{if } s_j = n_j \text{ for all } j < i \\ s_i & \text{otherwise.} \end{cases}$$

Equipping S_k with the product topology, each α_k is a minimal homeomorphism of S_k (for $k \neq \infty$, there is only one orbit of α_k). In the case $k = \infty$, the dynamical system $(S_\infty, \alpha_\infty)$ is called the *generalized adding machine* described by the list (n_i). Distinct lists can give rise to conjugate adding machines [8, 9].

It will be convenient in what follows to say that n (circular) subdisks $D_1, \ldots, D_n$ of a (circular) disk D are placed *regularly* in D if they are mutually disjoint, all have the same radius, and have centers equidistant from the center of D and at angles $2\pi j/n$. (In particular, this means that their radii are less than half that of D). Let $D = D^2$ be the unit disk, and place n_1 disks, labeled D_s for $s \in S_1$, regularly in D. In each of these n_1 disks D_s, place regularly n_2 disks labeled D_t, where $t \in S_2$ and $p(t) = s$. Proceed inductively: at the ith stage of the construction, for each $s \in S_{i-1}$, place regularly in D_s n_i disks labeled D_t, where $t \in S_i$ and $p(t) = s$. Since the diameters of the disks tend to zero as $i \to \infty$, it follows that

$$A = \bigcap_{k \geq 1} \bigcup_{s \in S_k} D_s$$

is a Cantor set, and there is a natural homeomorphism $h: A \to S_\infty$ given by $h(x) = (s_1, s_2, \ldots)$, where $x \in D_{(s_1, \ldots, s_k)}$ for each k.

For each finite allowable sequence $s \in S_k$, let C_s be an annular collar of the boundary of D_s, chosen in such a way that all of these collars are mutually disjoint. Let $\hat{D}_s$ be the closure of the disk D_s minus the disks and collars inside it,

$$\hat{D}_s = \mathrm{cl}(D_s \setminus \bigcup_{t: p(t) = s} (D_t \cup C_t)).$$

Then define the k^{th} *level* of the disk collection to be

$$L_k = \bigcup_{s \in S_k} \hat{D}_s.$$

Figure 3 shows two stages of the construction with $n_1 = 3$ and $n_2 = 2$.

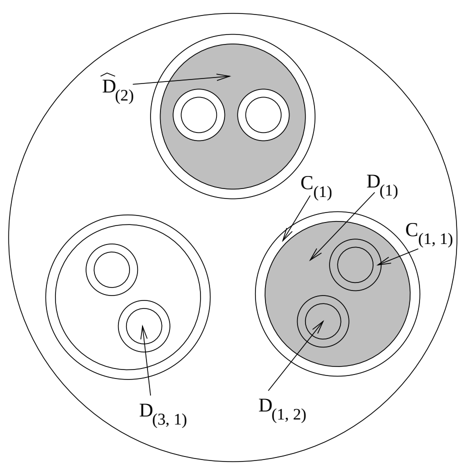

Figure 3: Initial stages of the construction of Φ with $n_1 = 3$ and $n_2 = 2$

Now define the homeomorphism $\Phi: D^2 \to D^2$ by defining it on each level L_k and on each of the collars. For each k, the restriction $\Phi: L_k \to L_k$ is required to satisfy $\Phi(\hat{D}_s) = \hat{D}_{\alpha_k(s)}$ for each $s \in S_k$. Thus the collection of $\hat{D}_s$ with $s \in S_k$ are cyclically permuted by Φ with period $m_k = \prod_{1 \leq i \leq k} n_i$. Hence Φ^{m_k} is a homeomorphism of each $\hat{D}_s$, which is required to be either pseudo-Anosov, or an adapted finite order map, with the latter being defined as follows.

If B is a disk with m regularly placed holes, then an adapted finite order map of B is a rigid rotation of angle $2\pi j/m$ for some j, followed by the time one map of the flow depicted in figure 4 for the case $m = 3$. Note that j must be chosen coprime to m, in order that the holes are cyclically permuted by the map. By making the time one map of the flow an irrational rotation on all of the boundary components of B, it can be arranged that the only periodic point is the fixed point at the center of B.

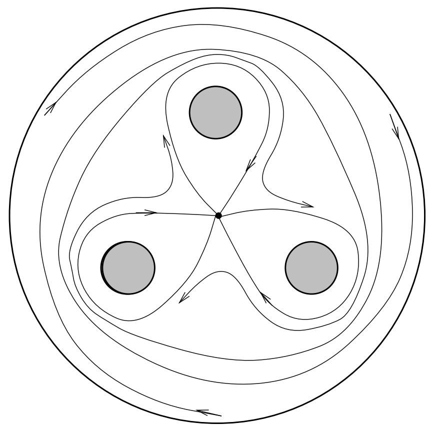

Figure 4: The flow used in the construction of an adapted finite order map

Now extend Φ over the collars C_s in such a way that there are no periodic points in the interiors of the collars (so for $s \in S_k$, Φ^{m_k} could push all orbits from one boundary of C_s to the other). No other constraint is placed on the behavior of Φ on the collars: up to isotopy, this is defined by the number of Dehn twists along their core circles.

Because the diameters of the disks tend to zero, there is a unique extension of Φ over A which makes it a homeomorphism of D^2, and the restriction $\Phi\colon A \to A$ is conjugate via $h\colon A \to S_\infty$ to the generalized adding machine $\alpha_\infty\colon S_\infty \to S_\infty$. Notice that the isotopy class of Φ relative to A is determined by the precise choice of adapted finite order or pseudo-Anosov maps on each level, and by the number of Dehn twists in the collars. In particular, there are uncountably many such choices for each conjugacy class of generalized adding machine.

The map Φ has one periodic orbit on each finite order level, and infinitely many periodic orbits on each pseudo-Anosov level. It will be shown that each of these periodic orbits is unremovable relative to A. Notice first that A contains no periodic points, and hence all of the periodic points are Nielsen bounded away from A (Remark 3.1 a)). Next, by an easy extension of the usual Nielsen-Thurston case (see [7] Section 1), no periodic point in the interior of a level L_k is in the same Nielsen class rel A as any other periodic point, under any iterate of Φ. By construction, these periodic points all have non-zero index.

Periodic points on the boundary of a level L_k cause slightly more difficulty. They may lie in the same Nielsen class as other points of the same orbit (on the same boundary component), and may not therefore be Nielsen separated: they are, however, irreducible (Remark 3.2 b): see [7] where irreducible is called "uncollapsible"). They may also lie in the same Nielsen class as periodic points on the other boundary of the collar C_s to which they belong. It is argued in [7] that in this case also, the total strong Nielsen class has non-zero index.

Thus if p is any periodic point of Φ, the periodic point collection (p, Φ) is strong Nielsen bounded, irreducible (and hence uncollapsible), and essential. It follows by Theorem 2.4 that (p, Φ) is unremovable. Thus any homeomorphism which is isotopic to Φ rel A has a periodic orbit of each braid type which is exhibited by Φ. By analogy with the Nielsen-Thurston theory, it would be possible to start with a description of the rel A isotopy class (say by its action on a nested family of circles), and then construct Φ as the dynamically minimal representative of the class. This point of view connects with the results of Bell and Meyer [2], who show that stable generalized adding machines embedded in plane homeomorphisms are always the limit of periodic points. In the class of examples presented here, all of these periodic points are unremovable relative to the minimal set. The precise nature of the periodic points depends on the isotopy class relative to A.

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