Title: Planes, Nets and Webs

Abstract:

A (2-dimensional) $k$-web has point set $W \subset \mathbb{R}^2$, an open neighbourhood of $0$. It has $k$ smooth coordinate functions $x_1, x_2, \ldots, x_k : W \to \mathbb{R}$ such that for all $i \neq j$, $\nabla x_i$ and $\nabla x_j$ are linearly independent throughout $W$; also $x_i(0) = 0$.

The level curves for $x_1, x_2, \ldots, x_k$ intersect transversely, forming the ‘lines’ of the web. Point $P \in W$ has $k$ coordinates $x_1(P), x_2(P), \ldots, x_k(P)$, any two of which uniquely determine the point $P$. Two webs are the same if they agree in a neighbourhood of $0$ (so only the germs of the coordinate functions $x_i$ are relevant).

Consider the vector space $V$ consisting of all $k$-tuples $(f_1, f_2, \ldots, f_k)$ of smooth functions $f_i : \mathbb{R} \to \mathbb{R}$ such that $f_i(0) = 0$ and

$$f_1(x_1(P)) + \cdots + f_k(x_k(P)) = 0$$

for all $P \in W$. The rank of $W$ is $\dim V \leq \frac{1}{2}(k-1)(k-2)$ (cf. Lie, Poincaré, Chern and Griffiths). Some examples attaining this upper bound are obtained from plane curves of degree $k$ having maximal genus (so-called extremal curves).

We are interested in finite analogues of webs, as follows. A $k$-net of prime order $p$ has $p^2$ points and $pk$ lines. Each line has $p$ points. There are $k$ parallel classes of lines, and each parallel class partitions the points. We must have $k \leq p+1$; and a $(p+1)$-net is the same thing as an affine plane of order $p$. Affine planes of prime order exist, by a classical construction using finite fields. A celebrated open problem asks if non-classical planes of prime order also exist. I have conjectured that the rank bound, stated above for webs, also holds for nets. The validity of this conjecture would imply that non-classical planes of prime order cannot exist.