UF Math Colloquium

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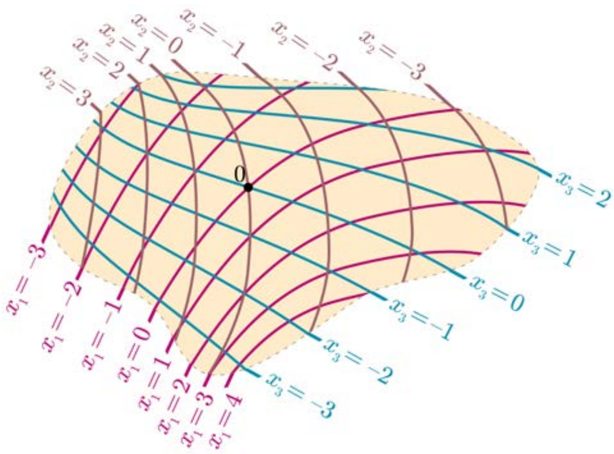
Title: Planes, Nets and Webs

Abstract:

A (2-dimensional) *k-web* has point set $\mathcal{W} \subset \mathbb{R}^2$, an open neighbourhood of $\mathbf{0}$. It has k smooth coordinate functions $x_1, x_2, \ldots, x_k : \mathcal{W} \to \mathbb{R}$ such that for all $i \neq j$, ∇x_i and ∇x_j are linearly independent throughout $\mathcal{W}$; also $x_i(\mathbf{0}) = 0$.



The level curves for $x_1, x_2, \ldots, x_k$ intersect transversely, forming the 'lines' of the web. Point $P \in \mathcal{W}$ has k coordinates $x_1(P), x_2(P), \ldots, x_k(P)$, any two of which uniquely determine the point P. Two webs *are the same* if they agree in a neighbourhood of $\mathbf{0}$ (so only the germs of the coordinate functions x_i are relevant).

Consider the vector space $\mathcal{V}$ consisting of all k-tuples $(f_1, f_2, \ldots, f_k)$ of smooth functions $f_i : \mathbb{R} \to \mathbb{R}$ such that $f_i(0) = 0$ and

$$f_1(x_1(P)) + \cdots + f_k(x_k(P)) = 0$$

for all $P \in \mathcal{W}$. The *rank* of $\mathcal{W}$ is $\dim \mathcal{V} \leqslant \frac{1}{2}(k-1)(k-2)$ (cf. Lie, Poincaré, Chern and Griffiths). Some examples attaining this upper bound are obtained from plane curves of degree k having maximal genus (so-called *extremal* curves).

We are interested in finite analogues of webs, as follows. A *k-net of prime order p* has p^2 points and pk lines. Each line has p points. There are k parallel classes of lines, and each parallel class partitions the points. We must have $k \leqslant p+1$; and a $(p+1)$-net is the same thing as an *affine plane* of order p. Affine planes of prime order exist, by a classical construction using finite fields. A celebrated open problem asks if non-classical planes of prime order also exist. I have conjectured that the rank bound, stated above for webs, also holds for nets. The validity of this conjecture would imply that non-classical planes of prime order cannot exist.