HW2 – Due Friday, February 3, start of class
No electronic submissions, only hard copy
Be sure to justify all your answers completely.

1. (*10 points*) For $a > 0$ let
$$g(x) = \frac{x^3 + 3ax}{3x^2 + a}. \tag{1}$$
   (a) Show that the fixed points of g are $0, \pm\sqrt{a}$.
   (b) Show that g is locally convergent at $\sqrt{a}$.
   (c) Show that g has a cubic rate of convergence at $\sqrt{a}$.
   (d) Compute the asymptotic constant
$$M = \lim_{n\to\infty} \frac{|\sqrt{a} - x_{n+1}|}{|\sqrt{a} - x_n|^3} \tag{2}$$
   where $x_n = g^n(x_0)$ and x_0 is chosen close enough to $\sqrt{a}$ so that $x_n \to \sqrt{a}$. Hint: use formulas (1) and (2) directly and *not* the third derivative.

2. Let $f(x) = \cos(x) + \sin^2(50x)$.
   (a) Write a computer program to implement Newton's method for f.
   (b) By doing numerical experiments, find out how close to the root $p = \pi/2$ an initial point x_0 needs to be in order to have the Newton iterates x_n converge to $p = \pi/2$.
   (c) Explain your results.

3. Let
$$G(x, y) = \left(\frac{.25}{1 + (x + y)^2}, \frac{.25}{1 + (x - y)^2} \right).$$
   Find a convex region B with $G(B) \subset B$ and $\|DG\|_\infty \leq k < 1$ for some k for all $(x, y) \in B$. Conclude then that G has a unique fixed point in B.

4. Let $f(x) = \log(x)$, $x_0 = 1, x_1 = 1.75, x_2 = 2$. Find the *total* error bound for the degree two interpolating polynomial $P_2(x)$ with these nodes and f on the interval $[1, 2]$, *i.e.* find a K with
$$\max_{t\in[1,2]} |f(t) - P_2(t)| \leq K.$$