HW3 – Due Friday, February 17, start of class
No electronic submissions, only hard copy
Be sure to justify all your answers completely.

1. For \( a = x_0 < x_1 < \cdots < x_n = b \) and a function \( f \in C[a, b] \), show that the interpolation problem has a unique solution,

\[
Q(x) = \sum_{j=0}^{n} c_je^{jx}
\]

with \( Q(x_i) = f(y_i) \) for all \( i \). Hint: reduce to a usual polynomial interpolation problem.

2. Consider the inner product on \( C[1, 2] \),

\[
\langle f, g \rangle = \int_1^2 f(x)g(x)e^{-x} \, dx.
\]

(a) Starting with the basis \( \{1, x, x^2\} \) for \( P_2[1, 2] \) use Gram-Schmidt to determine the first three orthonormal polynomials on \( [1, 2] \) with respect to the inner product.

(b) Find the order 2 (i.e. quadratic) best least squares approximation for \( f(x) = e^x \) on \( [1, 2] \) with respect to the given inner product.

3. Let \( f(x) = 1/(1 + x^2) \) on \([-5, 5]\).

(a) Write a program which uses Newton divided differences to find the interpolating polynomial \( p \) to \( f \) with respect to \( n = 10 \) uniformly spaced nodes.

(b) Evaluate \( p(0.4835) \) and compute the absolute error \( |p(0.4835) - f(0.4835)| \).

4. If \( f \in C^{2n+2}[a, b] \) and \( x_0, \ldots, x_n \) are distinct points in \( [a, b] \) and \( H_{2n+1} \) is the corresponding Hermite polynomial, show that for \( t \in [a, b] \)

\[
f(t) = H_{2n+1}(t) + \frac{(t-x_0)^2 \cdots (t-x_n)^2}{(2n+2)!} f^{(2n+2)}(\eta)
\]

for some \( \eta \in [a, b] \). Hint: imitate the proof of the corresponding result for the Lagrange polynomial using \( E(x) = f(x) - H_{2n+1}(x) \), \( w(x) = (x-x_0) \cdots (x-x_n) \), and

\[
G(x) = E(x) - \frac{w(x)^2}{w(t)^2} E(t)
\]

5. Fix \( n \). For the basic Lagrange polynomials for \( k = 0, \ldots, n \)

\[
L_k(x) = \prod_{i \neq j} \frac{x - x_j}{x_i - x_j},
\]

show that for all \( x \)

\[
\sum_{j=0}^{n} L_j(x) = 1.
\]

Hint: don’t do brute force algebra.