**HW3 – Due Friday, February 17, start of class**
**No electronic submissions, only hard copy**
**Be sure to justify all your answers completely.**

1. For $a = x_0 < x_1 < \cdots < x_n = b$ and a function $f \in C[a,b]$, show that the intepolation problem has a unique solution,
$$Q(x) = \sum_{j=0}^{n} c_j e^{jx}$$
with $Q(x_i) = f(y_i)$ for all $i$. Hint: reduce to a usual polynomial interpolation problem.

2. Consider the inner product on $C[1,2]$,
$$\langle f, g\rangle = \int_1^2 f(x)g(x)e^{-x}\,dx.$$
   (a) Starting with the basis $\{1, x, x^2\}$ for $\mathcal{P}_2[1,2]$ use Gram-Schmidt to determine the first three orthonormal polynomials on $[1,2]$ with respect to the inner product.
   (b) Find the order 2 (i.e. quadratic) best least squares approximation for $f(x) = e^x$ on $[1,2]$ with respect to the given inner product.

3. Let $f(x) = 1/(1+x^2)$ on $[-5,5]$.
   (a) Write a program which uses Newton divided differences to find the interpolating polynomial $p$ to $f$ with respect to $n = 10$ uniformly spaced nodes.
   (b) Evaluate $p(.4835)$ and compute the absolute error $|p(.4835) - f(.4835)|$.

4. If $f \in C^{2n+2}[a,b]$ and $x_0, \ldots, x_n$ are distinct points in $[a,b]$ and $H_{2n+1}$ is the corresponding Hermite polynomial, show that for $t \in [a,b]$
$$f(t) = H_{2n+1}(t) + \frac{(t-x_0)^2 \ldots (t-x_n)^2}{(2n+2)!} f^{(2n+2)}(\eta)$$
for some $\eta \in [a,b]$. Hint: imitate the proof of the corresponding result for the Lagrange polynomial using $E(x) = f(x) - H_{2n+1}(x)$, $w(x) = (x-x_0)\ldots(x-x_n)$, and
$$G(x) = E(x) - \frac{w(x)^2}{w(t)^2}E(t)$$

5. Fix $n$. For the basic Lagrange polynomials for $k = 0, \ldots, n$
$$L_k(x) = \prod_{i \neq j} \frac{x - x_j}{x_i - x_j},$$
show that for all $x$
$$\sum_{j=0}^{n} L_j(x) = 1.$$
Hint: don't do brute force algebra.