HW4 – Due Friday, March 3, start of class
No electronic submissions, only hard copy
Be sure to justify all your answers completely.

1. Find the value of α which minimizes
$$\int_{-1}^{1} (x^2 - (x+\alpha))^2 \, dx.$$
Be sure to show that your solution is, in fact, the global minimum by using an additional test.

2. For each n, let $x_0^{(n)}, x_1^{(n)}, \ldots x_n^{(n)}$ be the zeros of the n^{th} order Chebychev polynomial T_n on $[-1,1]$ and $Q_n(x)$ is the degree n interpolating polynomial to e^x on $[-1,1]$ with the nodes $x_0^{(n)}, x_1^{(n)}, \ldots x_n^{(n)}$ Show that
$$\lim_{n\to\infty} \|Q_n - e^x\|_\infty = 0.$$

3. Find the second order (i.e. quadratic) least squares approximation to the function e^x on the interval $[1,3]$ with respect to the weight $\alpha(x) \equiv 1$ using the fact that the first three orthonormal polynomials on $[-1,1]$ with respect to the weight $\alpha(x) \equiv 1$ are
$$\varphi_0(x) = \sqrt{1/2}, \quad \varphi_1(x) = \sqrt{3/2}\, x, \text{ and } \varphi_2(x) = \frac{\sqrt{5}}{\sqrt{8}}(3x^2 - 1).$$

4. (a) Write a program that computes the linear least squares approximation for a data set $(x_1, y_1), \ldots, (x_m, y_m)$.
   (b) Generate 20 data points for $k = 1, \ldots, 20$ with $x_k = k/2$ and $y_k = 5x_k + 2 + \epsilon_k$ where ϵ_k is a random number uniformly distributed in $[-2, 2]$. In Matlab the following generates a length 20 column vector of such random numbers:
   `4 * rand(20,1) - 2 * ones(20,1)`
   (c) Use your program to find the best linear least squares fit to your data $(x_1, y_1), \ldots, (x_{20}, y_{20})$.
   (d) How close is your computed answer to the expected one? Why is the estimate of the slope so much better than that for the intercept?

5. Derive the central three point formula
$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{f^{(3)}(\eta)\, h^2}{6}$$
for some $\eta \in [x-h, x+h]$ using a pair of Taylor polynomials with remainders in the same fashion as the derivation of the three point formula for $f''(x)$ done in class.