These are additional problems to practise for the Numerical Linear Algebra Exam. Note that many of them are much easier than an exam problem and are the kind of things one would need to prove on the way to the solution of a larger problem.

1. (a) Define the spectral radius $\rho(A)$ for a square matrix $A$.
(b) Prove that $\rho\left(A^{k}\right)=\rho(A)^{k}$ for $k>0$.
(c) If $A$ is invertible, prove that $\rho\left(A^{-1}\right)=1 /\left|\lambda_{\text {min }}\right|$, where $\left|\lambda_{\min }\right|=\min \{|\lambda|: \lambda \in$ $\Lambda(A)\}$.
(d) Prove that $\rho(A) \leq\|A\|$ for any induced matrix norm on $A$.
(e) Prove that $\rho(A)$ is not a matrix norm.
2. If $A \in \mathbb{C}^{m, n}$ with $m \geq n$, prove that $A^{*} A$ is invertible if and only if $\operatorname{rank}(A)=n$.
3. If $A \in \mathbb{R}^{m, n}$ with $m \geq n, \operatorname{rank}(A)=n$ and $b \in \mathbb{R}^{n}$.
(a) Define the least squares solution to $A x=b$.
(b) Derive the normal equations for the least squares problem.
(c) Prove that the unique solution to the least squares problem is $\left(A^{T} A\right)^{-1} A^{T} b$.
(d) Describe how to solve the least squares problem using the QR decomposition of $A$.
4. Define a normal matrix and prove that the following are equivalent.
(a) $A$ is normal
(b) $\|A x\|_{2}=\left\|A^{*} x\right\|_{2}$ for every $x$.
(c) $A$ is unitarily diagonalizable.
5. Prove that a normal triangular matrix is diagonal.
6. Assume $A \in \mathbb{R}^{m, m}$
(a) Prove that $\langle x, y\rangle_{A}=x^{*} A y$ is an inner product on $\mathbb{R}^{m}$ if and only if $A$ is symmetric and positive definite
(b) Assume now that $A$ is symmetric and positive definite. If $x_{*}$ is the solution to $A x=b$ and $\left\{p_{1}, \ldots, p_{m}\right\}$ is an orthonormal basis for $\mathbb{R}^{m}$ with respect to $\langle,\rangle_{A}$ and $x_{*}=\sum c_{i} p_{i}$, give a formula for the $c_{i}$.
7. If $\kappa_{2}(A)$ is the condition number of the square, non-singular $A$ with respect to the two-norm, prove that

$$
\kappa_{2}(A)=\frac{\sigma_{1}}{\sigma_{m}} .
$$

where $\sigma_{1}$ and $\sigma_{m}$ are the largest and smallest singular values of $A$, respectively.
8. If both $A$ and $U$ are in $\mathbb{C}^{m, m}$ and $U$ is unitary, prove that $\|U A\|_{2}=\|A\|_{2}$ and $\|U A\|_{F}=$ $\|A\|_{F}$
9. Prove that $\|A\|_{2}=\left(\rho\left(A^{*} A\right)\right)^{1 / 2}=\sigma_{1}$, where $\sigma_{1}$ is the largest singular values of $A$.
10. If $A$ is Hermitian, postive definite prove that its singular values are the same as it eigenvalues.
11. Compute $\operatorname{det}\left(\lambda I-w w^{*}\right)$ when $\lambda \in \mathbb{C}, I$ is the $m \times m$ identity matrix and $w \in \mathbb{C}^{m}$.
12. If $P$ is a projector, prove that $\operatorname{null}(P) \cap \operatorname{range}(P)=\emptyset$.
13. If $q_{1}, \ldots q_{n}$ is an orthonornal basis for the subspace $V \subset \mathbb{C}^{m}$ with $m>n$, prove that the orthognal projector onto $V$ is $Q Q^{*}$, where $Q$ is the matrix whose columns are the $q_{j}$.
14. Let $D \in \mathbb{R}^{m, m}$ be a diagonal matrix with all distinct entries $d_{j}$ on the diagonal and $w \in \mathbb{R}^{m}$, prove that the eigenvalues of $A=D+w w^{*}$ are the roots of the equation

$$
f(\lambda)=1+\sum_{j=1}^{m} \frac{w_{j}^{2}}{d_{j}-\lambda}
$$

15. Given Cholesky decomposition of the of Hermition positive definite matrix $A=R^{*} R$, prove that $\|R\|_{2}=\left\|R^{*}\right\|_{2}=\|A\|_{2}^{1 / 2}$.
16. For $A \in \mathbb{R}^{m, m}$ and symmetric, define its Rayleigh quotient $R(x)$ and show that when $\lambda$ is an eigenvalue of $A$ with eigenvector $v$, then $R(v)=\lambda$ and $\nabla R(v)=0$.
17. Assume that $T$ is tridiagonal and symmetric wiith the diagonal entries given by $a_{j}$ for $j=1, \ldots, m$ and the super- and sub-diagonal entries by $b_{j}$ for $j=1 \ldots m-1$. Let $p_{k}$ be the characteristic polynomial of the $k \times k$ matrix in the upper left hand corner of A. Prove that

$$
p_{k}(x)=\left(a_{k}-x\right) p_{k-1}(x)-b_{k-1}^{2} p_{k-2}(x) .
$$

18. Recall that Arnoldi iteration computes the Hessenberg decomposition $A=Q H Q^{*}$ sequentially by columns.
(a) If these first $n$ columns of $Q$ are $q_{1}, \ldots, q_{n}$, show that

$$
A q_{n}=h_{1, n} q_{1}+\ldots h_{n, n} q_{n}+h_{n+1, n} q_{n+1} .
$$

(b) Now assume that the Arnoldi iteration starts with $q_{1}=b /\|b\|_{2}$. If the iteration doesn't terminate, show that $\left\langle q_{1}, q_{2}, \ldots, q_{n}\right\rangle=\left\langle b, A b, \ldots, A^{n-1} b\right\rangle$.
19. Prove that every square matrix $A$ has a Schur factorization.
20. Assume that $A$ is real and symmetric. Here is the shifted QR algorithm:

$$
\begin{aligned}
& \begin{array}{l}
A^{(0)}=A \\
\text { for } k=[1: n] \\
\quad \text { Pick shift } \mu^{(k)} \\
\quad Q^{(k)} R^{(k)}=A^{(k-1)}-\mu^{(k)} I \\
\quad A^{(k)}=R^{(k)} Q^{(k)}+\mu^{(k)} I \\
\text { end }
\end{array} \text {. }
\end{aligned}
$$

Let $Q_{k}=Q^{(1)} \ldots Q^{(k)}$ and $R_{k}=Q^{(k)} \ldots Q^{(1)}$. Show that $A^{(k)}=Q_{k}^{T} A Q_{k}$ and $\left(A-\mu^{(k)} I\right)\left(A-\mu^{(k-1)} I\right) \ldots\left(A-\mu^{(1)} I\right)=Q_{k} R_{k}$.

