# Positivity of polynomials in matrix and operator variables

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# Positive (commutative) polynomials

## A classical warm-up

Let $\underline{x} = (x_1, \ldots, x_d)$ be commuting variables. A polynomial $f \in \mathbb{R}[\underline{x}]$ is positive if $p(\underline{\alpha}) \geq 0$ for all $\alpha \in \mathbb{R}^d$.

Obvious examples: sums of squares (SOS)

$$p_1^2 + \cdots + p_\ell^2$$

for $p_i \in \mathbb{R}[\underline{x}]$.

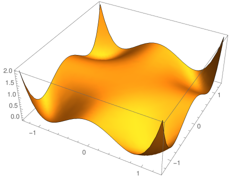
Gauss$^{1800}$: a positive univariate polynomial is a SOS.

# Hilbert's 17th problem

## Is every positive polynomial a SOS?

Hilbert[1888]: not true for $d > 1$.

Motzkin[65]: $x_1^4 x_2^2 + x_1^2 x_2^4 + 1 - 3x_1^2 x_2^2$
is positive but not SOS.



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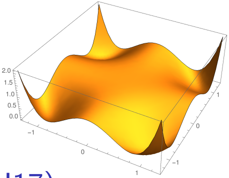


### Theorem (Artin 27; affirmative solution of H17)

*A polynomial $f$ is positive if and only if $p^2 \cdot f = s$ for some $p, s \in \mathbb{R}[\underline{x}]$ where $s$ is SOS.*

Back to Motzkin:

$$(x_1^2 + x_2^2)(x_1^4x_2^2 + x_1^2x_2^4 + 1 - 3x_1^2x_2^2)$$
$$= (x_1^2x_2^2(x_1^2 + x_2^2 + 1) + 1)(x_1^2 + x_2^2 - 2)^2$$

H17 was a breakthrough for **real** algebraic geometry.

# Real algebraic geometry

## Positivity when subject to constraints

RAG studies sets in $\mathbb{R}^d$ constrained by polynomial inequalities. To a set of constraints $C \subset \mathbb{R}[\underline{x}]$ we assign

$$\mathcal{K}_C = \{\underline{\alpha} \in \mathbb{R}^d \colon c(\underline{\alpha}) \geq 0 \text{ for all } c \in C\}$$

which is called a (basic closed) semialgebraic set if $C$ is finite.

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Polynomials that are obviously positive on $\mathcal{K}_C$:

$$s_0 + s_1 c_1 + \cdots + s_\ell c_\ell, \qquad c_i \in C, \; s_i \in \text{SOS}.$$

The set of such polynomials is called the quadratic module, $\mathbf{Q}_C$.

Alternative description of $\mathbf{Q}_C$: the smallest subset in $\mathbb{R}[\underline{x}]$ containing $C$ such that

$$1 \in \mathbf{Q}_C, \quad \mathbf{Q}_C + \mathbf{Q}_C \subseteq \mathbf{Q}_C, \quad p^2 \cdot \mathbf{Q}_C \subseteq \mathbf{Q}_C \text{ for } p \in \mathbb{R}[\underline{x}].$$

# Putinar's Positivstellensatz

## Probably the most used result from RAG

A quadratic module $\mathbf{Q}_C$ is Archimedean if there is $\rho > 0$ such that $\rho - x_i^2 \in \mathbf{Q}_C$ for $i = 1, \ldots, d$.

If $\mathbf{Q}_C$ is Archimedean, then $\mathcal{K}_C$ is bounded; if $\mathcal{K}_C$ is bounded, then we can add a constraint to $C$ to get an Archimeden quadratic module without changing $\mathcal{K}_C$.

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### Theorem (Putinar[93])

*Suppose $\mathbf{Q}_C$ is Archimedean. Then $f \geq 0$ on $\mathcal{K}_C$ if and only if $f + \varepsilon \in \mathbf{Q}_C$ for every $\varepsilon > 0$.*

Warning: $f + \varepsilon \in \mathbf{Q}_C$ for every $\varepsilon > 0$ does not imply $f \in \mathbf{Q}_C$.

# Polynomial optimization

## Plenty of packages in Matlab and Mathematica

Let $f \in \mathbb{R}[\underline{x}]$ and $C = \{c_1, \dots, c_\ell\} \subset \mathbb{R}[\underline{x}]$. Suppose $\mathbf{Q}_C$ is Archimedean ($\mathcal{K}_C$ is bounded).

Optimization problem: find $$\mu_* = \max\{f(\underline{\alpha})\colon \underline{\alpha} \in \mathcal{K}_C\}.$$

Equivalently, find $$\mu_* = \min\{\mu\colon \mu - f \geq 0 \text{ on } \mathcal{K}_C\}.$$

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Putinar: $\mu_* = \inf\{\mu\colon \mu - f \in \mathbf{Q}_C\}$.

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Relax: for $n \in \mathbb{N}$, find

$$\mu_n = \inf\{\mu\colon \mu - f = s_0 + s_1c_1 + \cdots + s_\ell c_\ell, \quad s_i \text{ SOS of deg} \leq 2n\}.$$

Then $\mu_n \searrow \mu_*$, and $\mu_n$ can be efficiently computed using semidefinite programming. (a generalization of linear programming)

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**Mantra**: checking positivity is a priori hard (geometry), checking for SOS is easy (algebra).

# Noncommutative positivity

| commutative | noncommutative |
|---|---|
| numbers | bounded operators on Hilbert spaces |
| reals | self-adjoint operators |
| $\geq 0$ | $\succeq 0$, positive semidefinite |
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Let $\underline{x} = (x_1, \ldots, x_d)$ be **free (noncommuting)** variables.
Elements of the free algebra $\mathbb{R}\langle\underline{x}\rangle$ are **free polynomials**, e.g.

$$3x_1x_2^2x_1x_2^2x_1 - x_2x_1^4x_2 + x_1x_2 + x_2x_1 - 2.$$

There is a natural involution $*$ on $\mathbb{R}\langle\underline{x}\rangle$ that fixes $x_j$:

$$(x_{i_1}x_{i_2}\cdots x_{i_\ell})^* = x_{i_\ell}\cdots x_{i_2}x_{i_1}.$$

# Three settings to consider

Free polynomials can be evaluated on tuples of operators on a Hilbert space. If $f = x_2 x_1^4 x_2 + x_1 x_2 + x_2 x_1 - 2$ and $\underline{X} \in \mathcal{B}(H)^2$,

$$f(\underline{X}) = X_2 X_1^4 X_2 + X_1 X_2 + X_2 X_1 - 2I \in \mathcal{B}(H).$$

Note: if $f = f^*$ and $\underline{X}$ is a tuple of self-adjoint operators, then $f(\underline{X})$ is self-adjoint.

We consider noncommutative analogs of global positivity (H17) and bounded positivity (Putinar) in three settings:

(i) positivity on $\mathrm{S}_n(\mathbb{R})^d$ for a **fixed** $n$

(ii) positivity on $\mathrm{S}_n(\mathbb{R})^d$ for **all** $n$

(iii) positivity on $\mathrm{S}(H)^d$, for a separable $\infty$-dim Hilbert space $H$

$\mathrm{S}_n(\mathbb{R})$ real symmetric $n \times n$ matrices, $\mathrm{S}(H)$ self-adjoint bounded operators

# Example

Let $g$ be the bivariate nc polynomial

$$2x_2x_1^3x_2 - x_2x_1x_2x_1^2 - x_1^2x_2x_1x_2 - x_1x_2x_1^2x_2 - x_2x_1^2x_2x_1 + x_1x_2^2x_1^2 + x_1^2x_2^2x_1$$

and let $f(x_1, x_2) := g(x_1^2, x_2)$.
So $f = f^* \in \mathbb{R}<\underline{x}>$, and we can talk about positivity of $f$.

Then

- $f(X_1, X_2)$ is positive semidefinite for every $2 \times 2$ symmetric matrices $X_1$ and $X_2$,
- $f(Y_1, Y_2)$ has a negative eigenvalue for

$$Y_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Thus $f \succeq 0$ on $S_2(\mathbb{R})^2$ (and $S_1(\mathbb{R})^2 = \mathbb{R}^2$), but $f \not\succeq 0$ on $S_3(\mathbb{R})^2$.

# Who cares?

## Sponsorship

- matrix inequalities in control theory
- free analysis
- polynomial optimization: noncommutative relaxations
- operator algebras and systems
- quantum information theory
- monotonicity (James) and convexity (Scott)

Background: courtesy of Bill Helton

# NC semialgebraic sets and quadratic modules

Sums of hermitian squares (SOHS):

$$p_1 p_1^* + \cdots + p_\ell p_\ell^*, \qquad p_i \in \mathbb{R}\langle \underline{x} \rangle .$$

To a subset of self-adjoint polynomials $C \subset \mathbb{R}\langle \underline{x} \rangle$ we assign

$$\mathcal{K}_C(n) = \{\underline{X} \in \mathrm{S}_n(\mathbb{R})^d : c(\underline{X}) \succeq 0 \text{ for all } c \in C\}$$

$$\mathcal{K}_C^{\mathrm{fin}} = \bigcup_{n \in \mathbb{N}} \mathcal{K}_C(n)$$

$$\mathcal{K}_C^{\infty} = \{\underline{X} \in \mathrm{S}(H)^d : c(\underline{X}) \succeq 0 \text{ for all } c \in C\}.$$

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The quadratic module $\mathbf{Q}_C$ generated by $C$ in $\mathbb{R}\langle \underline{x} \rangle$:

$$\sum_{i,j} p_{ij} c_i p_{ij}^*, \qquad c_i \in C \cup \{1\},\ p_{ij} \in \mathbb{R}\langle \underline{x} \rangle$$

That is, $\mathbf{Q}_C$ is the smallest subset of $\mathbb{R}\langle \underline{x} \rangle$ containing $C$ such that

$$1 \in \mathbf{Q}_C, \quad \mathbf{Q}_C + \mathbf{Q}_C \subseteq \mathbf{Q}_C, \quad p \cdot \mathbf{Q}_C \cdot p^* \subseteq \mathbf{Q}_C \text{ for } p \in \mathbb{R}\langle \underline{x} \rangle .$$

# Dimension-free results for free polynomials

## Global positivity

Theorem (McCullough[01], Helton[02])

*Let* $f = f^* \in \mathbb{R}\langle\underline{x}\rangle$. *Then* $f(\underline{X}) \succeq 0$ *for all* $\underline{X} \in \mathrm{S}_n(\mathbb{R})^d$ *and* $n \in \mathbb{N}$ *if and only if* $f$ *is SOHS.*

No denominators (cf. the classical H17) are needed!

Typical for dimension-free setting: you ask for more, you get more.

Given $f$, it actually suffices to check $f \succeq 0$ on $\mathrm{S}_n(\mathbb{R})^d$ for a large enough $n$ (depending on $d$ and $\deg f$).

# Dimension-free results for free polynomials

## Bounded positivity

$\mathbf{Q}_C$ Archimedean: $\rho - x_i^2 \in \mathbf{Q}_C$ for some $\rho > 0$.

### Theorem (Helton–McCullough[04])

*If $\mathbf{Q}_C$ is Archimedean, then $f \succeq 0$ on $\mathcal{K}_C^\infty$ if and only if $f + \varepsilon \in \mathbf{Q}_C$ for every $\varepsilon > 0$.*

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### Theorem (Helton–Klep–McCullough[12])

*If $\mathbf{Q}_C$ is Archimedean and $\mathcal{K}_C^{\text{fin}}$ is convex, then $f \succeq 0$ on $\mathcal{K}_C^{\text{fin}}$ if and only if $f \in \mathbf{Q}_C$.*

No $\varepsilon$ as in the classical Putinar is needed!

Didn't explain convexity... for later, $C = \{1 - x_1^2, \dots, 1 - x_d^2\}$

# Free polynomials, but fixed dimension

## Procesi–Schacher conjecture

### Conjecture (Procesi–Schacher[76])

*Fix $n$, and let $f = f^* \in \mathbb{R}<\underline{x}>$. Then $f(\underline{X}) \succeq 0$ for all $\underline{X} \in \mathrm{S}_n(\mathbb{R})^d$ if and only if*

$$pfp^* = s + p_0$$

*where $p, s, p_0 \in \mathbb{R}<\underline{x}>$, $s$ is a SOHS and $p_0$ vanishes on $\mathrm{S}_n(\mathbb{R})^d$.*

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Artin[27] (classical): true for $n = 1$

Procesi–Schacher[76]: true for $n = 2$

Klep–Špenko–V[18]: false for $n = 3$ — example of deg $= 15$

Educated guess: false for most $n$. Less sure for powers of 2; $n = 4$?

# Free polynomials, but fixed dimension

Bounded case is still nice

Theorem (Klep–Špenko–V[18])

*Fix n*, $f = f^* \in \mathbb{R}\langle \underline{x} \rangle$ *and* $C \subset \mathbb{R}\langle \underline{x} \rangle$. *If* $\mathbf{Q}_C$ *is Archimedean, then* $f \succeq 0$ *on* $\mathcal{K}_C(n)$ *if and only if for every* $\varepsilon > 0$,

$$f + \varepsilon - p_0 \in \mathbf{Q}_C$$

*for some* $p_0 \in \mathbb{R}\langle \underline{x} \rangle$ *vanishing on* $\mathrm{S}_n(\mathbb{R})^d$.

# Trace polynomials

Pure trace polynomials, $\mathbb{T}$, are polynomials in trace symbols $\operatorname{tr}(w)$ for words $w$ in $\underline{x}$, subject to the usual trace relations:

$$\operatorname{tr}(x_{i_1}x_{i_2}\cdots x_{i_\ell}) = \operatorname{tr}(x_{i_2}\cdots x_{i_\ell}x_{i_1}), \qquad \operatorname{tr}(w^*) = \operatorname{tr}(w).$$

So $\mathbb{T}$ is a polynomial ring in countably many generators $\operatorname{tr}(w)$, which are equivalence classes of words w.r.t. the "dihedral" action.

Trace polynomials: $\mathbb{T}\langle\underline{x}\rangle = \mathbb{T}\otimes\mathbb{R}\langle\underline{x}\rangle$.

$$\operatorname{tr}(x_1^2x_2x_1x_2) - \operatorname{tr}(x_1^2x_2)\operatorname{tr}(x_2) \ \in \mathbb{T},$$

$$\operatorname{tr}(x_1)x_1x_2x_1 - \operatorname{tr}(x_1^2x_2)\operatorname{tr}(x_1)\operatorname{tr}(x_2)x_2^2 + 2\operatorname{tr}(x_1^4) \ \in \mathbb{T}\langle\underline{x}\rangle.$$

# Trace polynomials continued

Originated in invariant theory.

Procesi[76]: every polynomial function $f : \mathrm{M}_n(\mathbb{R})^d \to \mathrm{M}_n(\mathbb{R})$ that is equivariant under simultaneous basis change, $f(P\underline{X}P^{-1}) = Pf(\underline{X})P^{-1}$, is given by a trace polynomial.

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We evaluate trace polynomials on tracial von Neumann algebras: pairs of a von Neumann algebra $\mathcal{F}$ (w.o.t.-closed unital $*$-subalgebra of bounded operators) and a faithful normal trace $\tau$ on $\mathcal{F}$.

E.g. $\mathrm{M}_n(\mathbb{R})$ with the *normalized* trace $\mathrm{tr}(A) = \frac{1}{n}\sum_{i=1}^{n} A_{ii}$.

# Traces and positivity

## traces of squares

(Sums of) hermitian squares and their traces (SOHST):

$$\operatorname{tr}(p_1 p_1^*)\cdots\operatorname{tr}(p_\ell p_\ell^*)p_0 p_0^*, \qquad p_i \in \mathbb{T}\langle\underline{x}\rangle\,.$$

To $C \subset \mathbb{T}\langle\underline{x}\rangle$ we assign $\mathcal{K}_C(n)$ and $\mathcal{K}_C^{\mathrm{fin}}$ as before; let $\mathcal{K}_C^{\mathrm{vna}}$ be the tuples satisfying the constraints $C$ from all tracial von Neumann algebras.

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The cyclic quadratic module $\mathbf{Q}_C$ is the smallest subset of $\mathbb{T}<\underline{x}>$ containing $C$ such that

$$1 \in \mathbf{Q}_C, \quad \mathbf{Q}_C + \mathbf{Q}_C \subseteq \mathbf{Q}_C, \quad p \cdot \mathbf{Q}_C \cdot p^* \subseteq \mathbf{Q}_C \text{ for } p \in \mathbb{T}<\underline{x}>$$

and

$$\operatorname{tr}(\mathbf{Q}_C) \subseteq \mathbf{Q}_C.$$

# Trace polynomials, fixed dimension

Theorem (Procesi–Schacher[76])

*Fix n, and let* $f = f^* \in \mathbb{T}{<}\underline{x}{>}$. *Then* $f(\underline{X}) \succeq 0$ *for all* $\underline{X} \in \mathsf{S}_n(\mathbb{R})^d$ *if and only if*

$$pfp^* = s + p_0$$

*where* $p, s, p_0 \in \mathbb{T}{<}\underline{x}{>}$, *s is a SOHST and* $p_0$ *vanishes on* $\mathsf{S}_n(\mathbb{R})^d$.

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Theorem (Klep–Špenko–V[18])

*Fix $n$, $f = f^* \in \mathbb{T}<\underline{x}>$ and $C \subset \mathbb{T}<\underline{x}>$. If $\mathbf{Q}_C$ is Archimedean, then $f \succeq 0$ on $\mathcal{K}_C(n)$ if and only if for every $\varepsilon > 0$,*

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*for some $p_0 \in \mathbb{T}<\underline{x}>$ vanishing on $\mathrm{S}_n(\mathbb{R})^d$.*

# What about the dimension-free setting?

The Connes[76] embedding conjecture for von Neumann algebras

(Kirchberg's conjecture for tensor products of $C^*$-alg, Tsirelson's problem in QIT)

has been recently refuted using complexity theory

(Ji–Natarajan–Vidick–Wright–Yuen[20]).

Theorem (Klep–Schweighofer[08])

*Let $d \geq 2$ and $C = \{1 - x_1^2, \ldots, 1 - x_d^2\}$. The failure of CEC is equivalent to the existence of $f \in \mathbb{R}\langle \underline{x} \rangle$ such that*

- $\mathrm{tr}(f(\underline{X})) \geq 0$ *for all* $\underline{X} \in \mathcal{K}_C^{\mathrm{fin}}$;
- $\mathrm{tr}(f(\underline{Y})) < 0$ *for some* $\underline{Y} \in \mathcal{K}_C^{\mathrm{vna}}$.

Conclusion: a quadratic module certificate can only work in the $\infty$-dimensional setting (even convex $\mathcal{K}_C^{\mathrm{fin}}$ doesn't help).

# The bounded setting and von Neumann algebras

## Theorem (Klep–Magron–V[20])

*Let $f \in \mathbb{T}$ and let $\mathbf{Q}_C \subset \mathbb{T}{<}\underline{x}{>}$ be Archimedean. Then $f \succeq 0$ on $\mathcal{K}_C^{\mathrm{vna}}$ if and only if $f + \varepsilon \in \mathbf{Q}_C$ for every $\varepsilon > 0$.*

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This version fails for $f \in \mathbb{T}<\underline{x}>$.

E.g. take $f = x_1$ and $C = \{1 - x_1^2\} \cup \{\mathrm{tr}(x_1 pp^*) : p \in \mathbb{R}<\underline{x}>\}$.

Not sure about $|C| < \infty$?

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### Theorem (Klep–Magron–V[20])

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This version fails for $f \in \mathbb{T}<\underline{x}>$.

E.g. take $f = x_1$ and $C = \{1 - x_1^2\} \cup \{\mathrm{tr}(x_1 pp^*) : p \in \mathbb{R}<\underline{x}>\}$.

Not sure about $|C| < \infty$?

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$$f = s_1(f) - s_2(f) \quad \textit{and} \quad \varepsilon - \mathrm{tr}(s_2(f)) \in \mathbf{Q}_C.$$

# Global trace positivity?

**Open questions** for $f \in \mathbb{T}{<}\underline{x}{>}$:

(i) Is $f \succeq 0$ on $\mathcal{K}_\emptyset^{\text{fin}}$ equivalent to $f \succeq 0$ on $\mathcal{K}_\emptyset^{\text{vna}}$?

(ii) Can at least the second one be certified using SOHST in some way?

(iii) If not, what is missing?

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Resolved in case $d = 1$, i.e., $\mathbb{T}{<}\underline{x}{>} = \mathbb{R}[x, \mathrm{tr}(x), \mathrm{tr}(x^2), \dots]$

Theorem (Klep–Pascoe–V[20])

*Let $d = 1$ and $f \in \mathbb{T}{<}\underline{x}{>}$. Then $f(X) \succeq 0$ for all $X \in \mathrm{S}_n(\mathbb{R})$ and $n \in \mathbb{N}$ if and only if $p^2 \cdot f = s$ where $p, s \in \mathbb{T}{<}\underline{x}{>}$ and $s$ is SOHST.*

# Summary

Free polynomials, $\mathbb{R}<\underline{x}>$

| | global | bounded |
|---|---|---|
| fixed $n$ | ✓/×/? [1] | ✓ |
| all $n$ | ✓ | ×/✓ [2] |
| $\infty$ | ✓ | ✓ |

Trace polynomials, $\mathbb{T}<\underline{x}>$

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|---|---|---|
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| all $n$ | ? [3] | × [4] |
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[1]: ✓ for $n = 1, 2$; × for $n = 3$; powers of two?

[2]: ✓ when convex; × in general

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# Thank you!