51

(1) $\nabla \Phi = [6x_1^2 + x_2^2 + 10x_1,\ 2x_1x_2 + 2x_2] = \vec{0}$

2nd eq 1st $0 = x_1 x_2 + x_2 = x_2(x_1+1) \Rightarrow x_2 = 0 \quad x_1 = -1$

1st eq $\quad 0 = 6x_1^2 + x_2^2 + 10x_1$, if $x_2 = 0,\ x_1 = 0$

, if $x_1 = -1 \quad x_2 = \pm 2$

(2) $H\Phi = \begin{bmatrix} 12x_1 + 10 & 2x_2 \\ 2x_2 & 2x_1 + 2 \end{bmatrix}$

$H\Phi(0,0) = \begin{bmatrix} 10 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad \lambda = 2, 10$ loc min

$H(-1,2) = \begin{bmatrix} -2 & 4 \\ +4 & 0 \end{bmatrix}$ $\quad \lambda = \dfrac{-2 \pm \sqrt{4 - 4(-16)}}{2}$

$= \dfrac{-2 \pm \sqrt{68}}{2}$ one pos, one neg, saddle

$H(-1,-2) = \begin{bmatrix} -2 & -4 \\ -4 & 0 \end{bmatrix}$ same e.values so saddle.

52

2. $\begin{bmatrix} 2 & -3 & 0 \\ 4 & -5 & 1 \\ 2 & -1 & 3 \end{bmatrix} \xrightarrow[-R1+R3 \to R3]{-2R1+R2 \to R2} \begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix} \xrightarrow{-2R2+R3 \to R3} \begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U$

so $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

1st solve $L\vec{y} = b$, $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \vec{y} = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$

$y_1 = 3$
$2y_1 + y_2 = 7, \; y_2 = 1$
$y_1 + 2y_2 + y_3 = 5 \; y_3 = 0$

now solve $U\vec{x} = \vec{y}$ $\begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

$2x_1 - 3x_2 = 3, \quad x_1 = 3$
$x_2 + x_3 = 1, \quad x_2 = 1$
$x_3 = 0$

$$\boxed{\vec{x} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}}$$

(4) False Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{null}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

$AA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\text{null}(AA) = \mathbb{R}^2$.

(3) $\text{Null}(A) = \{\vec{v} : A\vec{v} = \vec{0}\}$. If $\vec{v}, \vec{w} \in \text{Null}(A)$ then $\vec{0} = A\vec{v} = A\vec{w}$ and $A(\alpha\vec{v} + \beta\vec{w}) = \alpha A\vec{v} + \beta A\vec{w} = \vec{0} + \vec{0} = \vec{0}$ and so $\alpha\vec{v} + \beta\vec{w} \in \text{Null}(A)$ which is therefore a subspace.

(5) Split $\vec{v}$ into $\begin{bmatrix} \vec{x} \\ \vec{y} \\ \vec{z} \end{bmatrix}$ each $\vec{x}, \vec{y}, \vec{z}$ $(n \times 1)$ vectors

Then $B\vec{v} = [A\,A\,A] \begin{bmatrix} \vec{x} \\ \vec{y} \\ \vec{z} \end{bmatrix} = A\vec{x} + A\vec{y} + A\vec{z}$

$= A(\vec{x} + \vec{y} + \vec{z})$. Since $\text{Null}(A) = \{\vec{0}\}$ if

$0 = B\vec{v}$ then $\vec{x} + \vec{y} + \vec{z} = 0$. So Null(B) is

all vectors $\vec{v}$ of the form $\begin{bmatrix} \vec{x} \\ \vec{y} \\ \vec{z} \end{bmatrix}$ with $\vec{x} + \vec{y} + \vec{z} = 0$.

(6) We first compute eigen values and eigen vectors.

$\begin{vmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{vmatrix} = \lambda^2 - \lambda - 2$ so $\lambda = 2, -1$

$\lambda = -1$ yield $5v_1 - 5v_2 = 0$ so let $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 2$ yields $2v_1 - 5v_2 = 0$ so let $\vec{v} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

So $X = \begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix}$ and $X^{-1} \begin{bmatrix} 8 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

So in matrix form

$$\vec{x}(t) = \begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{10t} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

In vector Form

$$\vec{x}(t) = 3e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{10t} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

[54]

(8) $\begin{vmatrix} 11-\lambda & 6 \\ -15 & -7-\lambda \end{vmatrix} = (11-\lambda)(-7-\lambda)+90$
$= \lambda^2 - 4\lambda + 13$

$$\lambda = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

$\lambda = 2+3i$ $\quad 11-(2+3i)v_1 + 6v_2 = 0$
or $(9-3i)v_1 + 6v_2 = 0$ $\quad \vec{v} = \begin{bmatrix} -2 \\ 3-i \end{bmatrix}$
or $(3-i)v_1 + 2v_2$

$\lambda = 2-3i$ $\quad 11-(2-3i)v_1 + 6v_2 = 0$ $\quad \vec{v} = \begin{bmatrix} -2 \\ 3+i \end{bmatrix}$
or $(9+3i)v_1 + 6v_2 = 0$
or $(3+i)v_1 + 2v_2 = 0$
or

(9) A square matrix is orthogonal if $A^T = A^{-1}$

$$I = A^{-1}A = A^T A = \begin{bmatrix} \vec{c}_1^T \\ \vdots \\ \vec{c}_n^T \end{bmatrix} \begin{bmatrix} \vec{c}_1 & \cdots & \vec{c}_n \end{bmatrix}$$

$$= i \rightarrow \begin{bmatrix} & & \\ & \vec{c}_i^T c_j & \\ & & \end{bmatrix} = i \rightarrow \begin{bmatrix} & & \\ & \vec{c}_i \cdot \vec{c}_j & \\ & & \end{bmatrix}$$

$\uparrow$ j $\quad\quad \uparrow$ j

So $\vec{c}_i \cdot \vec{c}_j = \delta_{ij}$