Review soln - Exam #2 - FALL 17

(1) $\operatorname{Res}_{z=i} f = e^{2i}[1-2i]$ $\quad \operatorname{Res}_{z=0} f = -1$

(a) $2\pi i(-1)$ (b) $2\pi i[e^{2i}(1-2i)]$ (c) 0

(d) $2\pi i[-1+e^{2i}(1-2i)]$

(2) $\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1-2z}{z(1+3z)}$ which has $\operatorname{Res}_{z=0} = 1$

so $\oint \frac{z-2}{z(z+3)} = 2\pi i$

(3) $\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}$ $\quad |z|<1$

Take deriv $\frac{-2z}{(1+z^2)^2} = \sum_{n=1}^{\infty} (-1)^n (2n) z^{2n-1}$

and opposite $\frac{2z}{(1+z^2)^2} = \sum_{n=1}^{\infty} (-1)^{n+1} (2n) z^{2n-1}$ $\quad |z|<1$

(4) $\frac{1}{1+w} = \sum_{n=0}^{\infty} (-1)^n w^n$

$\operatorname{Log}(z) - \operatorname{Log}(1) = \int_1^z \frac{1}{1+w}\, dw = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{n+1} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}$

$\parallel$

$\operatorname{Log}(z)$

(5) $\frac{1}{(z-2)(z-3)} = -\frac{1}{z-2} + \frac{1}{z-3}$

(a) $\frac{1}{2-z} - \frac{1}{3-z} = \frac{1}{2}\frac{1}{1-\frac{z}{2}} - \frac{1}{3}\frac{1}{1-\frac{z}{3}} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$

(b) $\frac{1}{2-z} - \frac{1}{3-z} = -\frac{1}{z}\frac{1}{1-\frac{2}{z}} - \frac{1}{3}\frac{1}{1-\frac{z}{3}}$

$$= -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n - \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

(c) $\frac{1}{2-z} - \frac{1}{3-z} = -\frac{1}{z}\frac{1}{1-\frac{2}{z}} + \frac{1}{z}\frac{1}{1-\frac{3}{z}}$

$$= -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n + \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n$$

6) $-z - z^2 - \frac{5}{6}z^3 + \cdots$

7) (a) $\frac{1}{z} - \frac{1}{2} + \frac{1}{12}z + \cdots$

(b) $z=0$ is simple pole with Res $= 1$

so $\oint_{|z|=1} \frac{dz}{e^z - 1} = 2\pi i$

8) $z^2 e^{1/z} = z^2 + z + \frac{1}{2} + \frac{1}{6}z + \cdots$ so

$\operatorname{Res}_{z=0} z^2 e^{1/z} = 1/6$ and $\oint_{|z|=3} z^2 e^{1/z}\,dz = \frac{\pi i}{3}$

(9) $z = \pm\pi$ are simple poles because $\cos(\pm\pi) \neq 0$

$$\operatorname*{Res}_{z=\pi} \frac{\cos z}{(z-\pi)(z+\pi)} = \frac{\cos(\pi)}{2\pi} = -\frac{1}{2\pi}$$

$$\operatorname*{Res}_{z=-\pi} \frac{\cos z}{(z-\pi)(z+\pi)} = \frac{\cos(-\pi)}{-2\pi} = +\frac{1}{2\pi}$$

(a) $= -\frac{2\pi i}{2\pi} = i$ (b) $= 0$

(10) $$\frac{\operatorname{Log} z}{(z^2+1)^2} = \frac{\operatorname{Log} z/(z+i)^2}{(z-i)^2} = \frac{\phi(z)}{(z-i)^2}$$

Since $\phi(i) \neq 0$, $z = i$ is an order 2 pole

and $\operatorname*{Res}_{z=i} \frac{\operatorname{Log} z}{(z^2+1)^2} = \phi'(i) = \frac{\pi + 2i}{8}$

So the integral is $2\pi i\left[\frac{\pi+2i}{8}\right] = \frac{\pi^2 i}{4} - \frac{\pi}{2}$

(11) $\cot z = \frac{\cos z}{\sin z} = \frac{p(z)}{q(z)}$ has simple poles at

πn since $\cos \pi n \neq 0$ and $q'(\pi n) = \cos \pi n \neq 0$

So $$\operatorname*{Res}_{z=\pi n} \cot z = \frac{p(\pi n)}{q'(\pi n)} = \frac{\cos \pi n}{\cos \pi n} = 1$$

$-\pi, 0, \pi$ are inside $|z| = 4$ so integral $= 6\pi i$

(12) $\frac{\pi}{2\sqrt{2}}$ (13) $\frac{\pi}{2} e^{-5}$