REVIEW FOR EXAM 3 • SPRING 2020 • PROF. BOYLAND

You should know how to do all "show that" and hand computation from the HW's 9, 10, 11, 12. There are two parts of the review. Part I are problems some of which will appear verbatim on the exam. I will not give solutions for them, that is your job. Part II problems are more computational and solutions are provided.

Part I

1. State and prove the Convolution Theorem for the DFT.
2. If $\{\vec{q}_1, \ldots, \vec{q}_N\}$ is an orthonormal basis of $\mathbb{C}^N$ and $\vec{v} = \sum \alpha_i \vec{q}_i$, show that $\alpha_k = \langle \vec{q}_k, \vec{v}_k \rangle$.
3. (a) Define unitary matrix and show that if U is unitary then $\langle U\vec{x}, U\vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle$ for any two vectors $\vec{x}, \vec{y} \in \mathbb{C}^N$.

   (b) If $\hat{x} = \mathrm{DFT}(x)$, show that
   $$\sum_{j=0}^{N-1} |x_j|^2 = \sum_{j=0}^{N-1} |\hat{x}_j|^2.$$
   You may use the Pythagorean Theorem.
4. Define Hermitian matrix and show that a Hermitian matrix always has real eigenvalues.
5. If A is 100×100, B is 100×100, and C is 100×1, which order of multiplication is more efficient $(AB)C$ or $A(BC)$? Be sure to justify your answer completely.
6. If x is in $\mathbb{R}^N$ with N even and $\hat{x} = \mathrm{DFT}(x)$, show that for $k = 1, \ldots, \frac{N}{2} - 1$
   $$\overline{\hat{x}_k} = \hat{x}_{N-k}.$$

Part II

1. Let $f(x_1, x_2) = (x_1^2 x_2, x_1 x_2^3, x_1 x_2)$, $g(y_1, y_2, y_3) = (y_1 y_2, y_2 y_3)$, and $h = g \circ f$. Use the matrix chain rule to compute the matrix derivative Dh. Your answer must be in terms of x_1, x_2 and contain no y_1, y_2 or y_3 and be a single matrix.

2. Let $\overrightarrow{one} = [1, 1, \ldots, 1]^T$, the vector consisting of N ones. Compute $\mathrm{DFT}(\overrightarrow{one})$.

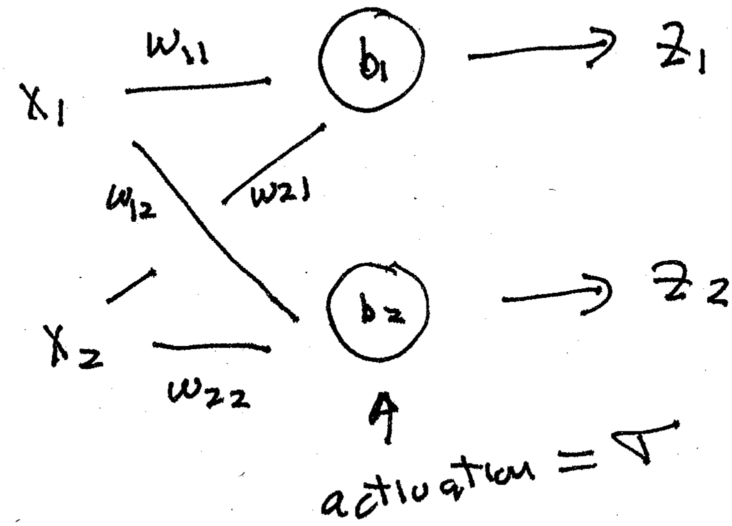
3. Given $u = [1, 3, -2]^T$ and $v = [1, 2, 3]^T$, by hand compute the cyclic convolution $u * v$. Show all work.

4. On $\mathbb{R}^2$ define the inner product
$$\langle \vec{u}, \vec{v} \rangle = \vec{u}^T \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \vec{v}$$
   (a) Starting with the basis $[1, 0]^T, [0, 1]^T$, using the Gram-Schmidt process compute an orthonormal basis with respect to this inner product.
   (b) Express the vector $\vec{w} = [-1, 1]^T$ in terms of your orthonormal basis.

5. Let $N = 5$.
   (a) Find the impulse response $\vec{g}$ so that for any vector $\vec{f}$,
$$(\vec{f} * \vec{g})_n = \frac{1}{2} f_{n-1} - \frac{1}{2} f_n + \frac{1}{2} f_{n+1}.$$
   (b) Give the 5×5 matrix M so that for any vector $\vec{f}$ we have $M\vec{f} = \vec{f} * \vec{g}$.

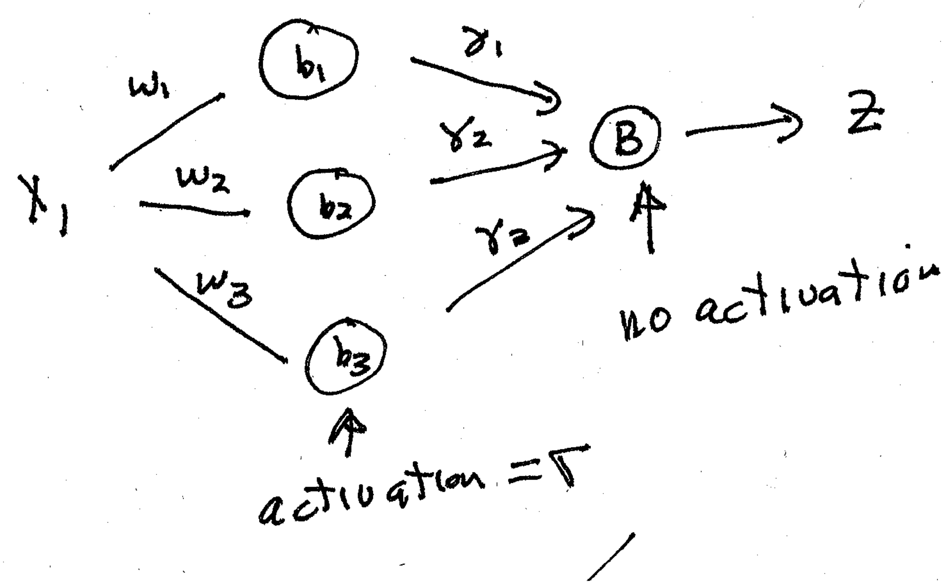
6. (A) For each of the net diagrams on the last page write down the form of the input-output function $F(x, \eta)$.
   (B) Assuming the activation σ is differentiable, for nets (a) and (b) write a formula for the derivative of F with respect to the parameters η.
   (C) For net (c), find the parameters w_1, w_2, b so that the network classifies $(1, 0)$, $(1, -1)$ and $(0, -2)$ as 0 and $(0, 0), (-1, 0)$ and $(0, 1)$ as 1.
   (D) For net (a) if the parameters are
$$W = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$
   and the activation is the ramp, σ_R, evaluate the output for the given input of $\vec{x} = [1, -1]^T$.

(a)



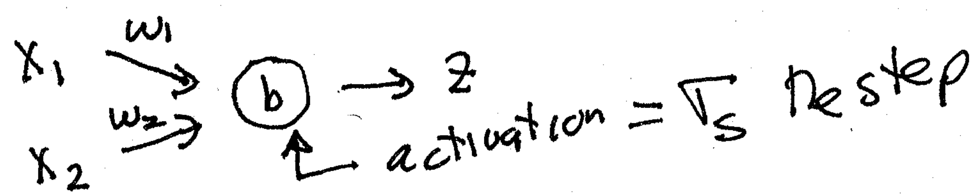
w_{11}
x_1
b_1
z_1
w_{12}
w_{21}
x_2
b_2
z_2
w_{22}
activation = σ

(b)



w_1
b_1
γ_1
x_1
w_2
b_2
γ_2
B
Z
γ_3
w_3
b_3
no activation
activation = σ

(c)



x_1
w_1
b
z
w_2
x_2
activation = σ_s the step.