New Dynamical Invariants on Hyperbolic Manifolds

Philip Boyland
Department of Mathematics
University of Florida
Gainesville, FL 32611-8105
boyland@math.ufl.edu

Abstract. The rotation measure is an asymptotic dynamical invariant assigned to a typical point of a flow in a fiber bundle over a hyperbolic manifold. The total mass of the rotation measure is the average speed of the orbit and its "direction" is the ergodic invariant probability measure of the hyperbolic geodesic flow which best captures the asymptotic dynamics of the given point. The rotation measure exists almost everywhere and is constant for an ergodic measure of the given flow and so it may be viewed as assigning an ergodic measure of the geodesic flow to one of the given flow. It generalizes the usual notion of homology rotation vector by encoding homotopy information.

Section 0: Introduction.

In a seminal paper Morse compared the geodesics of a general metric on a higher genus surface to the hyperbolic geodesics ([M]). His work when transfered into the language of dynamical systems says that there always exists a compact set invariant under the geodesic flow of the general metric which is semi-conjugate to the hyperbolic geodesic flow (cf. [DM]). The idea of comparing the dynamics of one system to a "canonical" one on the same manifold has continued to be fruitful and has found frequent application.

In this paper we compare the dynamics of a general flow ϕ_t on a bundle $\mathcal{B}$ over a closed hyperbolic manifold M to the dynamics of the hyperbolic geodesic flow g_t on the unit tangent bundle $T_1 M$ of the same manifold. This comparison makes use of the rotation measure which assigns to a typical point in the bundle an invariant measure of the hyperbolic geodesic flow. The assigned g_t-invariant measure gives, in a precise sense, the asymptotic direction of the ϕ_t-orbit through the point. The total mass of the rotation measure is the asymptotic progress of the lifted orbit in the universal cover. The first main theorem, Theorem 3.2, asserts that for an ergodic ϕ_t-invariant measure the rotation measure exists almost everywhere and is constant. Thus the rotation measure assigns a g_t-invariant measure $\rho(\eta)$ to a ϕ_t-invariant ergodic measure η. The second main theorem, Theorem 4.1, states that $\rho(\eta)$ is itself ergodic under g_t and further, that the system (ϕ_t, η) is measure theoretically semiconjugate to $(g_t, \rho(\eta))$.

Examples of dynamical systems where the rotation measure can be used are given in §1.1. These include flows on the hyperbolic manifold M itself, diffeomorphisms of M that are isotopic to the identity, surface diffeomorphisms which are isotopic to pseudoAnosov maps, and time periodic Euler-Lagrange flows whose configuration space is M. In the last case (which includes the geodesic flows of general metrics) it is known that there is an invariant measure whose rotation measure is equal to the Liouville measure of the geodesic

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flow (cf. [BG]). This is also the case for the surface diffeomorphism described in §5. Since the generic orbit for Liouville measure explores all the topology of the manifold and the geodesic flow is Bernoulli with respect to Liouville measure, its occurrence as a rotation measure indicates that the given system is dynamically very complicated. In general, a given flow will not have this level of complexity, and the rotation measures of various ergodic measures will be "smaller" measures within the geodesic flow. The dynamical complexity of the given flow is then quantified using the entropy and topology of these smaller rotation measures.

It is instructive to compare the rotation measure with the homology rotation vector which goes back to Schwartzman ([Sc], see [Bd], Section 11 for a review, and §4.3 below for a precise definition). This dynamical invariant is often used to quantify dynamics using the ambient topology and can be defined by lifting the flow to the universal free Abelian cover of M. (For simplicity of exposition we now restrict to the case of a flow on M itself.) The displacement of a lifted trajectory after a time t is given by an element of the vector space $H_1(M;\mathbb{R})$. The homology rotation vector of the trajectory is the average value of these displacement vectors as $t \to \infty$, if the limit exists. The rotation measure may be viewed as a generalization of the homology rotation vector which keeps track of homotopy classes of trajectories rather than homology classes.

To compute homotopy information it is necessary to lift the dynamics to the universal cover. If there is an equivariant Riemannian metric in the universal cover which has a unique geodesic arc connecting each pair of points (as in the hyperbolic case considered here), then the amount of displacement of a trajectory after a time t is gauged by the geodesic arc from the initial point on the trajectory to its position after time t. An asymptotic average of these geodesic arcs must now be computed. The method employed here is to identify the arc with the arclength measure supported on it, and then take the asymptotic average of these measures using the weak limit.

The rotation measure can also be defined using approximating loops and closed geodesics. Given a long trajectory in M, glue on a small arc to close it into a loop (cf. [Fd]). If one was computing the homology rotation vector, the homology class of this loop would be divided by the elapsed time, and then the limit taken as $t \to \infty$. To compute the rotation measure, pass to the unique, closed, hyperbolic geodesic in the same free homotopy class as the loop, take the arc length measure on the closed geodesic divided by the elapsed time, and then take the weak limit as $t \to \infty$. This makes it clear that a periodic orbit of the given flow ϕ_t has a rotation measure supported on the unique closed geodesic in its free homotopy class. The total mass of the rotation measure will be the speed of the projection of the orbit onto the closed geodesic. Thus, in particular, the rotation measure distinguishes periodic orbits that are in the same homology classes but different homotopy classes.

An essential first step in the proof of the main theorems is provided by Lemma 2.2. It states that for a positive progress, ergodic, ϕ_t-invariant measure almost every point is such that its lifted orbit in the Poincaré ball has unique limit points in forward and backward time on the sphere at infinity, and these limit points are distinct. This allows the almost everywhere definition of a shadowing geodesic which has the same limits on the sphere at infinity as the trajectory. The shadowing geodesics provide the basis for the comparison of the dynamics of the given measure to dynamics within the geodesic flow. The semiconjugacy in Theorem 4.1 is induced by projection onto shadowing geodesics.

Section 5 contains an example in which shadowing geodesics exist almost everywhere as required by Theorem 3.2 but they do not exist for a topologically generic point. This shows the necessity of working in the measure theoretic category, even though the dynamics are smooth. The example also shows that, in general, trajectories are not a bounded distance from their shadowing geodesic; this distance is always $o(t)$ by Lemma 2.3.

Section 1: Preliminaries.

This section contains various definitions and results required in this paper. In many cases these are stated only for the situation here, rather than in their greatest generality. The reader is urged to consult the references for proofs and further information.

§**1.1 Spaces and dynamics.** Throughout this paper M is a closed hyperbolic manifold, i.e. M is compact without boundary and has a Riemannian metric g on M with curvature identically equal to -1. The universal cover of a n-dimensional hyperbolic manifold is diffeomorphic to the standard $\mathbb{R}^n$. When $\mathbb{R}^n$ is equipped with a metric of constant curvature -1 it is denoted $\mathbb{H}^n$. Depending on the context we will use $\tilde{M}$ or $\mathbb{H}^n$ to denote the universal cover of M. The group of covering transformations in $\tilde{M}$ can be identified with a discrete subgroup, isomorphic to $\pi_1(M)$, of the group of isometrics of $\mathbb{H}$. This group action has a fundamental domain with compact closure and under the quotient by the action the metric on $\mathbb{H}$ descends to the hyperbolic metric on M.

The geodesic flow of the metric g is denoted g_t and is defined on T_1M the unit tangent bundle of the manifold. As a consequence of the Mostow Rigidity Theorem, a closed hyperbolic manifold of dimension three or greater carries a unique hyperbolic metric. If M is a surface, it carries many hyperbolic metrics, but their geodesic flows are all topologically conjugate ([G], [M]).

We shall usually work in a smooth locally trivial fiber bundle $p : \mathcal{B} \to M$ with fiber F. Letting $\tilde{\mathcal{B}} = F \times \tilde{M}$ yields a bundle $p : \tilde{\mathcal{B}} \to \tilde{M}$ which covers our original bundle, but in general is not the universal cover of $\mathcal{B}$.

There is another projection of importance, from covering spaces to bases; $\pi : \tilde{M} \to M$ and $\pi : \tilde{\mathcal{B}} \to \mathcal{B}$. For a set $Z \subset \mathcal{B}$, a tilde indicates its total lift, so $\tilde{Z} = \pi^{-1}(Z)$. On the other hand, a single point in $\tilde{\mathcal{B}}$ will often be denoted $\tilde{z}$, and the convention is that $\pi(\tilde{z}) = z$.

The main dynamical object here is a C^1-flow ϕ_t on $\mathcal{B}$, i.e. a flow generated by a continuous vector field on $\mathcal{B}$. This flow lifts to a flow $\tilde{\phi}_t$ on $\tilde{\mathcal{B}}$. For a closed hyperbolic manifold M examples of such flows are:

(1) $\mathcal{B} = M$, so that ϕ_t is a flow on M.
(2) $\mathcal{B} = T_*M$ or $\mathcal{B} = T^*M$ and ϕ_t is a Euler-Lagrange or Hamiltonian flow, respectively. A related case is $\mathcal{B} = T_*M \times S^1$ or $\mathcal{B} = T^*M \times S^1$ and the flow is induced by a time periodic Lagrangian or Hamiltonian (see [BG]).
(3) If $f : M \to M$ is a diffeomorphism isotopic to the identity, the flow ϕ_t is the suspension flow, and $\mathcal{B}$ is the suspension manifold. Since f is isotopic to the identity, $\mathcal{B}$ is diffeomorphic to $M \times S^1$ and the bundle $p : \mathcal{B} \to M$ is projection on the first factor.
(4) A related example is when N is a surface and $f : N \to N$ is a diffeomorphism in a pseudoAnosov isotopy class. In this case ϕ_t is the suspension flow. The suspension

manifold M has a hyperbolic structure by a theorem of Thurston, and so (M, ϕ_t) is as in example (1).

§**1.2 Hyperbolic Geometry.** For proofs and related information see [Be], [T], or [BKS]. We shall use two standard models built in Euclidean space for n-dimensional hyperbolic space $\mathbb{H}^n$. In both cases $d(z_1, z_2)$ will denote the *hyperbolic* distance between the two points, and $|z|$ will denote the usual *Euclidean* norm of the point z, i.e. $|z| = \langle z, z\rangle^{1/2}$, where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product. In addition, for a tangent vector v, $\|v\|_e$ denotes the norm in the tangent bundle induced by the Euclidean metric.

Given two distinct points $z, z' \in \mathbb{H}$, the notation $[z, z']$ indicates the unique oriented geodesic segment from z to z'. All geodesics in this paper are oriented and are considered both as subsets of the manifold M and its unit tangent bundle $T_1 M$ (as well as in $\tilde{M}$ and $T_1 \tilde{M}$). The distinction is usually unimportant, but when it is, it will be made explicit.

The *Poincaré disk model* is denoted $\mathbb{P}^n$ or just $\mathbb{P}$ if the dimension is left unspecified. The space in this case is the interior of the Euclidean unit n-ball and the hyperbolic metric induces a norm on the tangent bundle given by

$$\|v\|_h = \frac{2\|v\|_e}{1 - |z|^2}.$$

The origin is always denoted $\vec{\mathbf{0}}$. The unit $(n-1)$-sphere that is the Euclidean boundary of $\mathbb{P}^n$ is called the *sphere at infinity* and is denoted S_∞. A geodesic in this model is an oriented arc of a circle with both ends orthogonal to S_∞. Each pair of distinct points in S_∞ determine exactly one such geodesic and so the set of geodesics in parameterized by $(S^{n-1} \times S^{n-1}) - \{\text{diagonal}\}$. This space when endowed with the Euclidean topology and Lebesgue measure is denoted $\mathcal{G}$. In particular, a map defined on $\mathcal{G}$ is called measurable if it is Borel measurable. In the Poincaré disk model the bracket notation is extended to include $[x, x']$ denoting the geodesic connecting distinct points $x, x' \in S_\infty$.

In the *upper half space model* the space is $\mathbb{U}^n = \mathbb{R}^{n-1} \times (0, \infty)$ and it has coordinates $(\mathbf{x}, y)$ with $\mathbf{x} \in \mathbb{R}^{n-1}$. The norm on the tangent bundle is

$$\|v\|_h = \frac{\|v\|_e}{y}.$$

Geodesics are oriented lines and arcs of circles orthogonal to the boundary hyper-plane, i.e. to $\mathbb{R}^{n-1} \times \{0\}$.

The half space model is often convenient for explicit calculations. In analogy to the two-dimensional case, for a point $z \in \mathbb{U}^n$ with $z = (\mathbf{x}, y)$, $\bar{z} = (\mathbf{x}, -y)$ and $\mathrm{Im}(z) = y$. With these conventions:

$$\begin{aligned} d(z, w) &= \log\left(\frac{|z - \bar{w}| + |z - w|}{|z - \bar{w}| - |z - w|}\right) \\ \sinh\left(\frac{1}{2} d(z, w)\right) &= \frac{|z - w|}{2(\mathrm{Im}(z)\mathrm{Im}(w))^{1/2}} \qquad (1.1) \\ \cosh(d(z, w)) &= 1 + \frac{|z - w|^2}{2\mathrm{Im}(z)\mathrm{Im}(w)} \end{aligned}$$

The first lemma will be useful in comparing measures supported on different sides of a geodesic triangle. It is what one would expect from the "thin triangles" property in hyperbolic geometry. No attempt was made to optimize the estimate.

Lemma 1.1: *If $g_i : [0, \ell_i] \to \mathbb{H}$ for $i = 1, 2, 3$ with $\ell_i < \infty$ are parameterizations by arc length of the sides of a geodesic triangle with $\ell_1, \ell_2 > \ell_3$ and $\ell_1, \ell_2 > 1$, and $f : \mathbb{H} \to \mathbb{R}$ is a C^1-function with*

$$m = \sup\{|f(z)| + |D_z f(v)| : z \in \mathbb{H}, \|v\|_h = 1\} < \infty$$

then

$$\left| \int_0^{\ell_1} f(g_1(s))\, ds - \int_0^{\ell_2} f(g_2(u))\, du \right| \le m(5\,\ell_3 + 1).$$

Proof: The side of the triangle parameterized by g_i is called γ_i, the angle opposite γ_i is θ_i, and assume that $g_1(0) = g_2(0)$ is the vertex with angle θ_3. We may assume without loss of generality that $\ell_1 \ge \ell_2$, and so $\theta_1 \ge \theta_2 > \theta_3$.

Let $k(s)$ be the length of the geodesic segment that has one endpoint on γ_1 at the point $g_1(s)$, is orthogonal to γ_1, and the other end of the geodesic segment is on γ_2. Define $u(s)$ so that this point is $g_2(u(s))$. Note that $k(s) < \ell_3$ and an easy argument using hyperbolic trigonometry yields $d(u(s))/ds > 1$. Let $\hat{u} = \ell_2 - \ell_3$ and define $\hat{s}$ so that it satisfies $u(\hat{s}) = \hat{u}$, and let $\hat{k} = k(\hat{s})$.

Now

$$\begin{aligned}
&\int_0^{\ell_1} f(g_1(s))\, ds - \int_0^{\ell_2} f(g_2(u))\, du \\
&= \int_0^{\hat{s}} f(g_1(s)) - f(g_1(s))\frac{d(u(s))}{ds}\, ds + \int_0^{\hat{s}} f(g_1(s))\frac{d(u(s))}{ds} - f(g_2(u(s)))\frac{d(u(s))}{ds}\, ds \\
&\qquad + \int_{\hat{s}}^{\ell_1} f(g_1(s))\, ds - \int_{\hat{u}}^{\ell_2} f(g_2(u))\, du.
\end{aligned}$$

From left to right these integrals are called I_1, I_2, I_3, and I_4.

Because $d(u(s))/ds > 1$,

$$|I_1| \le m \int_0^{\hat{s}} \frac{d(u(s))}{ds} - 1\, ds = m(\hat{u} - \hat{s}) \le m\hat{k} \le m\ell_3,$$

where $\hat{u} - \hat{s} \le \hat{k}$ by the geodesic triangle inequality. The hyperbolic law of sines yields

$$k(u) < \sinh(k(u)) = \sinh(u)\sin(\theta_3) = \frac{\sinh(u)\sinh(\ell_3)}{\sinh(\ell_2)} \le e^{u + \ell_3 - \ell_2}$$

where in the last inequality we used the hypothesis $\ell_2 > 1$. Since f is Lipschitz with constant m on the geodesic segment from $g_1(s)$ to $g_2(u(s))$,

$$\begin{aligned}
|I_2| &\le m \int_0^{\hat{s}} k(s)\frac{d(u(s))}{ds}\, ds = m \int_0^{\hat{u}} k(u)\, du \\
&\le m(e^{\hat{u} + \ell_3 - \ell_2} - e^{\ell_3 - \ell_2}) = m(e^0 - e^{-\hat{u}}) \le m.
\end{aligned}$$

Finally,

$$
\begin{aligned}
|I_3 - I_4| &\leq |I_3| + |I_4| \\
&\leq m(\ell_1 - \hat{s} + \ell_2 - \hat{u}) \\
&\leq m(2\ell_3 + \hat{k} + \ell_3) \leq 4m\ell_3
\end{aligned}
$$

using the triangle inequality on the geodesic quadrilateral with vertices $g_1(\hat{s}), g_1(\ell_1), g_2(\ell_2)$, and $g_2(\hat{u})$. $\square$

§**1.3 Dynamical Cocycles.** For more information see [HK] or [Po]. The study of ergodic invariant measures is essential for understanding the dynamics of a flow ϕ_t. Even if the flow lives on a smooth manifold, restricting attention to an invariant measure requires the notion of a flow in the measure theoretic category.

A (measure theoretic) flow is a triple (Z, η, ϕ_t), consisting of a measure space Z, a measurable flow ϕ_t, and an invariant measure η. The flow is required to be continuous on orbits, i.e. the flow considered as a map $Z \times \mathbb{R} \to Z$ is measurable in the first component but *continuous* in the second. In the cases of interest here, Z will always have a natural topology and we shall always consider the σ-algebra as the Borel sets, so the σ-algebra is not included in the notation for a flow.

A function $C : Z \times \mathbb{R} \to \mathbb{R}$ is called a *additive cocycle* for the flow ϕ_t if

$$
C(z, s+t) = C(z, s) + C(\phi_s(z), t) \tag{1.2}
$$

and a *sub-additive cocycle* if

$$
C(z, s+t) \leq C(z, s) + C(\phi_s(z), t) \tag{1.3}
$$

for all $z \in Z$ and $s, t \in \mathbb{R}$. A cocycle is called *Lipschitz* with respect to the invariant measure η if is uniformly Lipschitz in the second variable on almost every trajectory. Using (1.2) or (1.3) this is equivalent to the existence of a constant $\kappa > 0$ so that for a.e. z, $|C(z,t)| \leq \kappa |t|$ for all t. Clearly a Lipschitz cocycle satisfies $C(z, 1) \in L^1(\eta)$ and so Theorem 1.2 below applies. Note that the Lipschitz condition is *only* required to hold in the t variable. In general, a Lipschitz cocycle $C(z,t)$ will depend only measurably on z. All the various geometric cocycles defined in this paper will turn out to be Lipschitz.

The forward average asymptotic value of a cocycle is

$$
C^*(z) = \lim_{t \to \infty} \frac{C(z,t)}{t}
$$

if the limit exists. We shall make frequent use of

Theorem 1.2: (Kingman's Sub-additive Ergodic Theorem) *If* (Z, η, ϕ_t) *is a flow with* η *ergodic, and* C *is a sub-additive cocycle for* ϕ_t *with* $C(z,1) \in L^1(\eta)$, *then* $C^*(z)$ *exists almost everywhere and has the constant value*

$$
C^*(\eta) = \inf_{t \in \mathbb{R}^+} \{ \frac{1}{t} \int C(z,t) \, d\eta(z) \} \tag{1.4}
$$

which in the case of an additive cocycle is equal to

$$C^*(\eta) = \int C(z,1)\, d\eta(z). \tag{1.4$'$}$$

Given a flow ϕ_t and a cocycle C, let Z_C denote the set of points for which C^* exists. Points in Z_C will be called ***generic*** for C. Another way of phrasing Kingman's theorem is that Z_C has full measure with respect to any ϕ_t-invariant measure for which $C(z,1)$ is in L^1. In the sequel we shall often be concerned with a fixed *ergodic* measure, in which case Z_C means just those elements of Z_C in the support of the measure.

Note that if C is a Lipschitz additive cocycle, then $C'(z) := \frac{\partial C}{\partial t}(z,0)$ exists a.e. and is bounded. Differentiating (1.2) with respect to t and evaluating at $t=0$ yields $C'(\phi_s(z)) = \frac{\partial C}{\partial t}(z,s)$ and so

$$C(z,t) = \int_0^t C'(\phi_s(z))\, ds.$$

Thus for Lipschitz additive cocycles Kingman's Theorem reduces to the Birkhoff ergodic theorem.

The asymptotics of cocycles in backwards time will also be needed in the sequel. Note that if C is a sub-additive cocycle for ϕ_t, then $\hat{C}(z,t) := C(z,-t)$ is one for $\psi_t = \phi_{-t}$. Call a sub-additive cocycle ***symmetric*** if $C(\phi_t(z),-t) = C(z,t)$ for all t. It follows easily from (1.3) that a symmetric cocycle is non-negative for all z and t. For a symmetric cocycle,

$$\int \hat{C}(z,t)\, d\eta(z) = \int C(z,-t)\, d\eta(z) = \int C(\phi_t(z),-t)\, d\eta(z) = \int C(z,t)\, d\eta(z) \tag{1.5}$$

where the second equality uses the fact that η is a ϕ_t-invariant measure. Thus dividing by t and using (1.4) yields $\hat{C}^*(\eta) = C^*(\eta)$ for an ergodic η. Thus for an ergodic measure η, a generic z, and a symmetric cocycle C, we may write

$$C(z,t) = C^*(\eta)\, |t| + o(t). \tag{1.6}$$

In contrast, for an additive cocycle one always has $C(\phi_t(z),-t) = -C(z,t)$ and so (1.5) with the appropriate sign changes shows that for an ergodic measure η, a generic z, and an additive cocycle C, we may write

$$C(z,t) = C^*(\eta)\, t + o(t). \tag{1.7}$$

It is important to note that for both additive and subadditive cocycles the exact form of the $o(t)$ term can depend strongly on the choice of the point z.

§**1.4 Measure-valued cocycles.** The definition of the rotation measure in §3 makes use of cocycles that take their value in a space of signed measures. Let X be a compact metric space and $\mathcal{M}(X)$ denotes the Banach space of all finite signed Borel measures on X. Recall that a sequence of measures $\mu_n \to \mu$ ***weakly*** if $\int f\ d\mu_n \to \int f\ d\mu$ for all continuous $f : X \to \mathbb{R}$. Given a flow (Z, ϕ_t, η) as in §1.3, a function $N : Z \times \mathbb{R} \to \mathcal{M}(X)$ is called a ***measure-valued cocycle*** for ϕ_t if for all $s, t \in \mathbb{R}$ and all $z \in Z$, $N(z, t+s) = N(z, t) + N(\phi_t(z), s)$ where the sum is the usual sum of signed Borel measures. As with other cocycles, we let

$$N^*(z) = \lim_{t\to\infty} \frac{N(z,t)}{t}$$

if the weak limit exists.

Given a function $f \in C(X, \mathbb{R})$, then $N_f(z, t) := \int f\ dN(z, t)$ is a real-valued additive cocycle. The existence of the asymptotic average N^* of a measure-valued cocycle given in the next lemma follows easily from the existence of the various N_f^* and the Riesz Representation Theorem. The lemma is clearly not the most general of its type in the literature. Let $\mathbf{1} : Z \to \mathbb{R}$ be the constant function $\mathbf{1}(z) = 1$ for all $z \in Z$.

Lemma 1.3: *If N is a measure-valued cocycle for ϕ_t and η is an ergodic, ϕ_t-invariant probability measure with $N_{\mathbf{1}}(z, 1) \in L^1(\eta)$, then N^* exists and is constant almost everywhere.*

Proof: First note that $N_{\mathbf{1}}(z, 1) \in L^1(\eta)$ implies $N_f(z, 1) \in L^1(\eta)$ for any $f \in C(T_1M, \mathbb{R})$, and so $N_f^*(\eta)$ exists by Theorem 1.2. Let Φ denote the real-valued linear functional $\Phi : f \mapsto N_f^*(\eta)$. Now if z is generic for both N_f and $N_{\mathbf{1}}$, then

$$\begin{aligned}\Phi(f) &= \lim_{t\to\infty} \frac{\int f\ dN(z,t)}{t}\\ &\le \|f\|_0 \lim_{t\to\infty} \frac{N_{\mathbf{1}}(z,t)}{t}\\ &= \|f\|_0 N_{\mathbf{1}}^*(\eta).\end{aligned}$$

Thus Φ is bounded and so by the Riesz Representation Theorem there is a measure $\hat{N} \in \mathcal{M}(X)$ with $\Phi(f) = \int f\ d\hat{N}$. Now pick a countable dense set $\{f_0 = \mathbf{1}, f_1, f_2, \ \ldots\}$ in $C(X, \mathbb{R})$ and let Z_N be the full measure set of points that are generic for all the corresponding cocycles, i.e.

$$Z_N = \cap Z_{N_{f_i}}.$$

So for $z \in Z_N$,

$$\frac{\int f_i\ dN(z,t)}{t} = \frac{N_{f_i}(z,t)}{t} \to \Phi(f_i) = \int f_i\ d\hat{N}$$

for all f_i and so $\frac{N(z,t)}{t} \to \hat{N}$ weakly. □

§1.5 Semiconjugacies, time changes and invertible cocycles. For more information see [HK], [Pa] section 5.1, or [CFS]. The main strategy of this paper is to compare a given flow on a hyperbolic manifold to the geodesic flow. One way this is accomplished is via a measure theoretic semi-conjugacy. Two flows (X, η, ϕ_t) and (Y, μ, h_s) (as defined in §1.3) are said to be ***semiconjugate*** if there is a measure-preserving surjection $f : X \to Y$ that takes orbits of ϕ_t to those of h_s preserving the direction of the flow, but not necessarily the time parameterization. Further, we require that f be continuous when restricted to orbits. Except in the case when the image orbit is periodic, this means that when restricted to an orbit in X, f is a homeomorphism onto an orbit in Y. Note however that f may take many orbits of ϕ_t to the same orbit of h_s. (The reader is cautioned that there are many variants of this definition in the literature going under a variety of names.)

There is a new flow $\hat{\phi}_s$ obtained by a time change of ϕ_t that is semiconjugate to h_s by a *time-preserving* semiconjugacy. It will be useful to describe this explicitly. Assume for simplicity that (X, η, ϕ_t) is aperiodic, i.e. the set of closed orbits has measure zero. Given the map $f : X \to Y$ as above, define $A(x,t)$ as the unique real number with the property that $h_{A(x,t)} \circ f(x) = f \circ \phi_t(x)$. It is easy to check that A is an additive cocycle and it is injective and onto in the second factor. Thus there is another additive cocycle $B(x,s)$ with $A(x, B(x,s)) = s$ and $B(x, A(x,t)) = t$. Now define a new flow on X by $\hat{\phi}_s(x) = \phi_{B(x,s)}(x)$, and then $h_s \circ f(x) = f \circ \hat{\phi}_s(x)$, for all s, x, thus h_s and $\hat{\phi}_s$ are semiconjugate by a time-preserving semiconjugacy.

The additive cocycle A can be used for a time change because for fixed z, $A(z, \cdot)$ is a homeomorphism $\mathbb{R} \to \mathbb{R}$. Such a cocycle will be called ***invertible***. Using (1.2) the injectivity of $A(z, \cdot)$ is equivalent to a monotonicity property. An additive cocycle A is called ***monotone***, if for all $\tau > 0$, $A(z, \tau) > 0$ (equivalently, $t > s$ implies $A(z,t) > A(z,s)$), and ***semi-monotone*** if the inequalities are not strict, i.e. $A(z, \tau) \geq 0$. Note that a monotone cocycle with $A^*(\eta) > 0$ is invertible.

If η is an ergodic probability measure for ϕ_t, and the time changed flow $\hat{\phi}_s$ is constructed using the invertible Lipschitz cocycle A, then the vector field that generates $\hat{\phi}_s$ is obtained by multiplying the generator of ϕ_t by $1/A'(z)$, where $A'(z) = \frac{\partial A}{\partial t}(z, 0)$. Further, the measure $\hat{\eta}$ defined by

$$d\hat{\eta} = \frac{A'(z)}{K}\, d\eta,$$

with $K = \int A'(z)\, d\eta$ is an ergodic, $\hat{\phi}_s$ invariant measure, and is, in fact, the only such measure that is equivalent to η.

In §4 a function arises that takes orbits to orbits as in a semiconjugacy, but is not locally injective on orbits and so does not give rise to a monotone cocycle A. However, it is the case that A is asymptotically monotone in the sense that $A^* > 0$. The next lemma says that we can alter A in a controlled fashion to obtain the required invertible cocycle. The alteration of A in (1.8) is usually expressed by saying that A is cohomologous to the alteration $\hat{A}$. As with Lemma 1.3 this is certainly not the most general result of its type in the literature.

Lemma 1.4: *If η is an ergodic, invariant probability measure for a flow ϕ_t on X and A is a Lipschitz, additive cocycle with $A^*(\eta) > 0$, then there exists a measurable $\beta : X \to (0, \infty)$ such that*

$$\hat{A}(x,t) := A(x,t) + \beta(\phi_t(x)) - \beta(x) \tag{1.8}$$

is an invertible, Lipschitz, additive cocycle for ϕ_t with $\hat{A}^*(\eta) = A^*(\eta)$.

Proof: We proceed in two steps, first producing a cocycle A_1 that is semi-monotone and then using it to produce $\hat{A}$.

Let $\beta_1(x) = \sup_{s\le 0} A(x,s)$. Since $A^* > 0$, β_1 is finite almost surely. Further, it is not difficult to check that β_1 is measurable, non-negative, continuous on orbits, and that

$$A_1(x,t) := A(x,t) + \beta_1(\phi_t(x)) - \beta_1(x)$$

is a semi-monotone additive cocycle.

Now let $\alpha(x,t) = \beta_1(\phi_t(x)) - \beta_1(x)$. We claim that α is a Lipschitz cocycle with the same constant as A, denoted κ. Property (1.2) is obvious, and since $\alpha(x,-t) = -\alpha(x,t)$ it suffices to assume $t > 0$. Since A is continuous on orbits, there are $s_1, s_2 \le 0$ with $\beta_1(\phi_t(x)) = A(\phi_t(x), s_1)$ and $\beta_1(x) = A(x, s_2)$, and so

$$\alpha(x,t) = A(\phi_t(x), s_1) - A(x, s_2).$$

Now if $t \ge -s_1$ then $|\alpha(x,t)| \le |A(\phi_t(x), s_1)| \le \kappa |s_1| \le \kappa t$. On the other hand, if $t < -s_1$, then using the definition of β_1, $s_2 = s_1 + t$, and since by (1.2), $A(\phi_t(x), s_1) = A(x, s_1+t) - A(x,t)$, we get $|\alpha(x,t)| = |A(x,t)| \le \kappa t$, proving the claim, and also proving that A_1 is Lipschitz.

Further, we claim that $\int \alpha(x,1)\ d\eta(x) = 0$ (this would be trivial if $\beta_1 \in \mathrm{L}^1(\eta)$, but that is not proven here). To prove the claim, for $n \in \mathbb{N}$, define $\beta_1^{(n)}$ as $\beta_1^{(n)}(x) = \beta_1(x)$, if $\beta_1(x) \le n$, and $\beta_1^{(n)}(x) = n$, otherwise, and let $\alpha^{(n)}(x,t) = \beta_1^{(n)}(\phi_t(x)) - \beta_1^{(n)}(x)$. Now certainly, $\beta_1^{(n)} \in L^1(\eta)$, and so by the invariance of the measure, $\int \alpha^{(n)}(x,1)\ d\eta(x) = 0$. Since $\alpha^{(n)} \to \alpha$ pointwise and $|\alpha(x,1)| < \kappa$, $\int \alpha(x,1) d\eta(x) = 0$ by the bounded convergence theorem. So using (1.4′), $A^*(\eta) = \hat{A}^*(\eta)$.

For the second step, pick $\psi : \mathbb{R} \to (0,\infty)$, supported on $[0,\infty)$ which satisfies $\int \psi = 1$ and $\int s\ \psi(s)\ ds = m < \infty$. Let $\beta_2(x) = \int_0^\infty A_1(x,s)\psi(s)\ ds$, and define

$$\begin{aligned}\hat{A}(x,t) &:= A_1(x,t) + \beta_2(\phi_t(x)) - \beta_2(x)\\ &= A_1(x,t) + \int A_1(\phi_t(x), s)\psi(s)\ ds - \int A_1(x,s)\psi(s)\ ds\\ &= \int (A_1(x, s+t) - A_1(x,s))\psi(s)\ ds\\ &= \int A_1(\phi_s(x), t)\psi(s)\ ds\end{aligned} \tag{1.9}$$

using cocycle property for A_1 and the fact that $\int \psi = 1$.

For $t > 0$, since A_1 is Lipschitz, (1.9) implies that $\hat{A}(x,t) \le \kappa t$, so $\hat{A}$ is Lipschitz also. Again using (1.9), for fixed $\tau > 0$, $\hat{A}(x,\tau) = \int A_1(\phi_s(x), \tau)\psi(s)\ ds$. The integrand is a continuous function of s which is nonnegative since A_1 is semi-monotone and positive somewhere for generic x since $A_1^*(\eta) > 0$, thus $\hat{A}(x,\tau) > 0$, and so $\hat{A}$ is monotone. Finally, since $|A_1(x,s)| \le \kappa |s|$, $|\beta_2| \le \kappa m$, thus $\beta_2 \in L^1(\eta)$. Thus $\int \beta_2(\phi_t(x)) - \beta_2(x)\ d\eta(x) = 0$, and so $\hat{A}^*(\eta) = A_1^*(\eta) = A^*(\eta)$. Since this is positive, $\hat{A}$ is invertible. □

Section 2: Geometric cocycles.

We now restrict attention to flows as in §1.1, namely smooth flows ϕ_t on a bundle above a closed hyperbolic manifold M. The motion of orbits in the universal cover are described using various geometric cocycles.

§2.1 The distance cocycle. The progress of lifted orbits in the universal cover is measured by a sub-additive cocycle, the *distance cocycle* (see [CF] and [K]). Given $z \in \mathcal{B}$, pick a lift $\tilde{z} \in \tilde{\mathcal{B}}$ and let

$$D(z,t) = d(p(\tilde{\phi}_t(\tilde{z})), p(\tilde{z})).$$

Note that this definition is independent of the choice of the lift $\tilde{z}$, and the sub-additive property of D is a direct consequence of the triangle inequality for the metric d.

If a ϕ_t-invariant measure η has compact support (as will always be assumed here), then because the flow is C^1, there is a bound, say κ, on the hyperbolic norm of the time derivative of a trajectory $\tilde{\phi}_t(z)$ when projected to $\mathbb{H}$. Thus $D(z,t) \leq \kappa|t|$ for every z in the support of η, and so D is Lipschitz with respect to η. Note also that D is symmetric as a consequence of the symmetric property of the metric. Thus by (1.6) for ergodic η and generic z we may write $D(z,t) = D^*(\eta)\ |t| + o(t)$.

The invariant measures of primary interest here represent dynamics where there is net average motion around the manifold, i.e. $D^*(\eta) > 0$. Such an ergodic measure will be said to have *positive progress* for ϕ_t.

Remark 2.1: There is another natural cocycle that measures speed in the cover, namely the length cocycle. If $\ell(x,t)$ denotes the hyperbolic arclength of the projected curve $p(\tilde{\phi}_{[0,t]}(\tilde{z}))$, then ℓ is clearly an additive cocycle bounded above by D. The asymptotic average ℓ^* is the average speed on the trajectory (where the norms of velocity vectors are taken using the hyperbolic metric). In general, ℓ^* can be strictly less than D^*. To distinguish the properties of an orbit measured by these two cocycles, D^* is described as the average *progress* of an orbit in the cover rather than the speed of the orbit.

§2.2 Limits on the sphere at infinity. The lemmas in this section show that positive progress measures have the property that generic orbits in the cover converge to points on the sphere at infinity. Further, these limits points are distinct as $t \to \infty$ and $t \to -\infty$. The convergence to a point at infinity turns out to require much weaker hypotheses than does the distinctness of the forward and backward limits. The first lemma gives only the existence of the limits. The idea for its proof came from Yair Minsky.

Lemma 2.1: *If $\gamma : [0,\infty) \to \mathbb{P}^n$ is a smooth path parameterized by arclength and $D(t) := d(\gamma(0), \gamma(t))$ is such that $\exp(-D(t))$ is integrable, then $\gamma(t) \to \omega \in S_\infty$.*

Proof: Since γ is parameterized by arclength, $|dD/dt| \leq 1$, and so the integrability assumptions imply that $D(t) \to \infty$ as $t \to \infty$. Thus there is a $T \geq 0$ such that $t \geq T$ implies $\gamma(t) \neq \vec{0}$.

Let $\beta : [T,\infty] \to S_\infty$ be the radial projection of $\gamma(t)$ on S_∞, *i.e.*

$$\beta(t) = \frac{\gamma(t)}{|\gamma(t)|}.$$

Computing one finds that

$$\langle \dot\beta(t), \dot\beta(t)\rangle = \frac{\langle \dot\gamma(t), \dot\gamma(t)\rangle - \frac{\langle \gamma(t), \dot\gamma(t)\rangle^2}{\langle \gamma(t), \gamma(t)\rangle}}{|\gamma(t)|^2}$$

and thus

$$\|\dot\beta(t)\|_e \leq \frac{\|\dot\gamma(t)\|_e}{|\gamma(t)|}.$$

A simple calculation using the metric on $\mathbb{P}^n$ yields $|\gamma(t)| = \tanh(D(z,t)/2)$ and by definition

$$\|\dot\gamma(t)\|_h = \frac{2\|\dot\gamma(t)\|_e}{1 - |\gamma(t)|^2},$$

and so

$$\|\dot\beta(t)\|_e \leq \frac{\|\dot\gamma(t)\|_h}{2\cosh(D(z,t)/2)\sinh(D(z,t)/2)} = \frac{2}{e^{D(t)} - e^{D(t)}}.$$

Thus the integrability assumptions on $D(t)$ imply that $\int_T^\infty \|\dot\beta(t)\|_e$ is finite and so $\beta([T,\infty)) \subset S_\infty$ has finite Euclidean length. This implies that there is a $\omega \in S_\infty$ with $\lim_{t\to\infty} \beta(t) = \omega$. Since $D(t) \to \infty$, $\lim_{t\to\infty} \gamma(t) = \omega$. $\square$

If $\tilde\phi_t(\tilde z)$ is a trajectory of the lifted flow on $\tilde{\mathcal{B}}$ and $p(\tilde\phi_t(\tilde z)) \to \omega \in S_\infty$ then define $\omega(\tilde z) = \omega$. Similarly, define $\alpha(\tilde z)$ as the limit as $t \to -\infty$ if that exists. If $\alpha(\tilde z)$ and $\omega(\tilde z)$ exist and are distinct, then the oriented geodesic $[\alpha(\tilde z), \omega(\tilde z)]$ is called the ***shadowing geodesic*** of the trajectory and is denoted $\Gamma_{\tilde z}$ (cf. [H]). The next lemma says that for an ergodic, positive progress measure for ϕ_t, the shadowing geodesic exists almost everywhere.

Lemma 2.2: *If η is a ϕ_t-invariant probability measure that is ergodic, has compact support, and $D^*(\eta) > 0$, then there exists a set $Z_1 \subset \mathcal{B}$ of full η-measure so that for all $\tilde z \in \tilde Z_1$, $\alpha(\tilde z)$ and $\omega(\tilde z)$ exist and are distinct. Further, the maps $\alpha, \omega : \tilde Z_1 \to S_\infty$ are measurable.*

Proof: Fix $\tilde z \in \tilde Z_D$ and let $\gamma(t) = p(\tilde\phi_t(\tilde z))$. If $\hat\gamma(s)$ is a reparametrization of γ by arc length and $\hat D(s) = d(\hat\gamma(0), \hat\gamma(s))$, then

$$\int_0^\infty \exp(-2\hat D(s))\, ds = \int_0^\infty \|\dot\gamma(t)\|_h \exp(-2D(z,t))\, dt$$

is finite because $D(z,t) = D^*(\eta)|t| + o(t)$ from §2.1 and $\|\dot\gamma(t)\|_h$ is bounded because γ is C^1 and η has compact support. Thus by Lemmas 2.1, $\omega(\tilde z)$ exists, and similarly, $\alpha(\tilde z)$ does also. The functions α and ω are measurable because they are constructed as the a.e. in z limit as $t \to \infty$ of a function continuous in $\tilde z$ and t. It remains to show that $\alpha(\tilde z) \neq \omega(\tilde z)$ for typical z. For this another cocycle is required.

Fix $x \in S_\infty$. The family of horospheres tangent to S_∞ at x can be given a parameterization $H_x(r)$ with $r \in \mathbb{R}$ which has the property that $d(H_x(r), H_x(s)) = |r - s|$. As a normalization assume that $H_x(0)$ contains the origin and as $r \to \infty$, the horospheres converge to x. If $y \in H_x(r)$ the Busemann function based at x is defined as $\Upsilon_x(y) = r$

(this definition is slightly non-standard, see [Bu]). Standard properties of the horosphere family yield that $d(y, H_x(s)) = |r - s| = |\Upsilon_x(y) - s|$, and this distance is realized by a geodesic segment that is orthogonal to both $H_x(r)$ and $H_x(s)$.

Now define

$$C(z,t) = \Upsilon_{\omega(\tilde{z})}(p(\tilde{\phi}_t(\tilde{z}))) - \Upsilon_{\omega(\tilde{z})}(p(\tilde{z})),$$

where $\tilde{z}$ is some choice of a lift of z. Clearly C is an additive cocycle for ϕ_t which measures the progress of orbits in the direction normal to the horospheres. In addition, the definition is independent of the choice of lift $\tilde{z}$ and the standard horosphere properties yield that $|C(z,t)| \leq D(z,t)$ for all t, and so C is a Lipschitz cocycle and Theorem 1.2 applies.

Let $Z_1 = Z_D \cap Z_C$. For $z \in Z_1$ by (1.7) $D(z,t) = c_1|t| + o(t)$ and $C(z,t) = c_2 t + o(t)$ where by assumption $c_1 > 0$. We next establish the claim $c_2 = c_1$ using the upper half space model, $\mathbb{U}$. Fix a $z \in \tilde{Z}_1$ and use an isometry that sends $\mathbb{P}$ to $\mathbb{U}$, $p(\tilde{z})$ to $(\vec{\mathbf{0}}, 1)$, and $\omega(\tilde{z})$ to ∞.

Let $\gamma(t) = p(\tilde{\phi}_t(\tilde{z}))$. By construction $\gamma(0) = (\vec{\mathbf{0}}, 1)$. Write the coordinates of $\gamma(t)$ as $(\mathbf{K}(t), H(t)) \in \mathbb{R}^{n-1} \times \mathbb{R}^+$. For the moment will suppress the dependence of $\mathbf{K}$, H and various cocycles on z and t. With this convention we have $C = (-1)^\delta d\big((\mathbf{K},1),(\mathbf{K},H)\big)$ with $\delta = 0$ when $H \geq 1$ and $\delta = 1$, otherwise, and $D = d\big((\vec{\mathbf{0}},1),(\mathbf{K},H)\big)$.

From the distance formulas (1.1),

$$H = \exp(C) = \exp(c_2 t + o(t)) \tag{2.1}$$

$$\cosh(D) = 1 + \frac{|\mathbf{K}|^2 + (H-1)^2}{2H} = \frac{|\mathbf{K}|^2}{2H} + \frac{H}{2} + \frac{1}{2H} \tag{2.2}$$

and the Cauchy-Schwarz inequality yields

$$\frac{1}{H}\frac{d|\mathbf{K}|}{dt} = \frac{1}{H}\frac{d\langle \mathbf{K},\mathbf{K}\rangle^{1/2}}{dt} = \frac{1}{H}\frac{\langle \mathbf{K},\dot{\mathbf{K}}\rangle}{\langle \mathbf{K},\mathbf{K}\rangle^{1/2}} \leq \frac{1}{H}\|\dot{\mathbf{K}}\|_e = \|\dot{\gamma}\|_h < \infty. \tag{2.3}$$

If $c_2 < 0$, then using (2.1) and (2.3), $\int_0^\infty \|\dot{\mathbf{K}}\|_e$ is finite. Thus, $H(t)$ is decreasing and $\mathbf{K}(t)$ is bounded, and so $\lim_{t\to\infty}\gamma(t)$ cannot be ∞ as assumed. On the other hand, if $c_2 \geq 0$, (2.3) implies $\limsup(1/t)|\mathbf{K}(t)| \leq c_2$, and so (2.2) gives $c_1 \leq 2c_2 - c_2 = c_2$. But since $|C(z,t)| \leq D(z,t)$, $|c_2| \leq c_1$, and so $c_2 = c_1 > 0$ as claimed.

To show that $\alpha(\tilde{z}) \neq \omega(\tilde{z}) = \infty$, we now have $c_2 > 0$ so (2.1) and (2.3) imply that $H(t) \to 0$ as $t \to -\infty$ and $\int_{-\infty}^0 \|\dot{\mathbf{K}}\|_e$ is finite, so $\lim_{t\to-\infty}\gamma(t) \neq \infty$. $\square$

Remark 2.2: The horocycle flow on hyperbolic surfaces makes clear the necessity of the positive progress hypothesis in order to get a shadowing geodesic. The horocycle flow has a unique ergodic invariant measure, call it ν. Using the upper half plane model $\mathbb{U}^2$, the members of the horocycle family of ∞ are the horizontal lines $y = c$. These yield generic trajectories of the horocycle flow that project to $\gamma(t) := (ct, c)$. The distance cocycle starting at $z = (0,c)$ is $D(t) = d((0,c),(ct,c)) = 2\log((\sqrt{t^2+4}+t)/2)$ using the formulas (1.1). Thus the hypothesis of Lemma 2.1 holds, and indeed α and ω both exist. However they are equal, which is in agreement with the fact that $D^*(\nu) = 0$ and so the hypotheses of Lemma 2.2 are not satisfied.

Remark 2.3: If a lifted trajectory is such that $\omega(x)$ exists then the orbit of $\omega(x)$ under the deck group as a subset of the sphere at infinity S_∞ is a topological invariant of the trajectory. This invariant has been studied in some detail for the case of flows on hyperbolic surfaces. See Section 6.2 of [ABZ] and the references therein.

§**2.3 Projection and asymptotics.** Lemma 2.2 says that the generic point of a positive progress measure has a shadowing geodesic. The next lemma describes the asymptotics of the distance from this geodesic and of the projection onto it. Given a geodesic and point in $\mathbb{H}$, hyperbolic orthogonal projection sends the point to a point on the geodesic. To get an image point in the unit tangent bundle define $\Sigma : \mathcal{G} \times \mathbb{H} \to T_1\mathbb{H}$ via $\Sigma(\Gamma, z) = (x, v)$ where x is the orthogonal projection of z onto Γ and v is the unit vector tangent to Γ at x. Note that Σ is continuous when $\mathcal{G}$ is given the topology described in §1.2.

Now fix a measure η as in Lemma 2.2. For $\tilde{z} \in \tilde{Z}_1$ from that lemma, let $\Gamma_{\tilde{z}}$ be the shadowing geodesic $[\alpha(\tilde{z}), \omega(\tilde{z})]$ and define $\sigma : \tilde{Z}_1 \to T_1\tilde{M}$ via $\sigma(\tilde{z}) = \Sigma(\Gamma_{\tilde{z}}, \tilde{z})$. Note that σ is equivariant (*i.e.*, it descends to a map $Z_1 \to T_1 M$ that is also called σ), is measurable (using Lemma 2.2), and takes orbits of $\tilde{\phi}_t$ to those of $\tilde{g}_t$.

We need two cocycles which are defined for $z \in Z_1$. Let $\Delta(\tilde{z}) = d\big(p(\tilde{z}), \Gamma_{\tilde{z}}\big) = d\big(p(\tilde{z}), p(\sigma(\tilde{z}))\big)$, and $B(z,t) = \Delta(\tilde{\phi}_t(\tilde{z})) - \Delta(\tilde{z})$. Thus B is an additive cocycle that measures the progress of the orbit though $\tilde{z}$ in a direction orthogonal to its shadowing geodesic. The projected progress onto the shadowing geodesic is measured by an additive cocycle A defined as follows. Fix a parameterization by arclength for each geodesic in $\mathbb{H}$. The parameterization is used to add and subtract elements on the geodesics. Given $z \in Z_1$ and $t \in \mathbb{R}$, let $A(z,t) = \sigma(\tilde{\phi}_t(z)) - \sigma(z)$, or equivalently, $A(z,t)$ is the unique $s \in \mathbb{R}$ with $\tilde{g}_s(\sigma(z)) = \sigma(\tilde{\phi}_t(z))$.

Note that both A and B are measurable. Further, since hyperbolic orthogonal projection onto a geodesic contracts tangent vectors, $|A(z,t)| \le D(z,t)$ for all z, t. By the triangle inequality, $B(z,t) \le D(z,t) + |A(z,t)|$. Thus since D is a Lipschitz cocycle, both A and B are also. The following proposition says that the rate of motion projected to the shadowing geodesic is the same as the progress of the motion and that the distance away from the shadowing geodesic grows at most like $o(t)$.

Lemma 2.3: *With η as in Lemma 2.2 and the cocycles A, B and D as above, $A^*(\eta) = D^*(\eta)$ and $B^*(\eta) = 0$.*

Proof: We will use the upper half space model, and let $Z_2 = Z_1 \cap Z_A \cap Z_B$ with Z_1 as in Lemma 2.2. Fix a $\tilde{z} \in \tilde{Z}_2$. Using an isometry we may arrange $\omega(\tilde{z}) = \infty$ and $\alpha(\tilde{z}) = \vec{\mathbf{0}}$, and so $\Gamma_{\tilde{z}} = \{\mathbf{x} = \vec{\mathbf{0}}\}$. We may also assume that $\gamma(0) = (\mathbf{L}, 1)$ for some $\mathbf{L} \in \mathbb{R}^{n-1}$. Call the coordinates of $\gamma(t) = (\mathbf{K}(t), H(t)) \in \mathbb{R}^{n-1} \times \mathbb{R}^+$. Note that $\omega(\tilde{z}) = \infty$ implies that $A(z,t) \to \infty$.

Again we suppress dependence of cocycles and coordinates on z and t. Hyperbolic orthogonal projection onto $\Gamma_{\tilde{z}}$ is denoted $\hat{\sigma}$. In Euclidean coordinates $\hat{\sigma}$ is particularly simple, $\hat{\sigma}(\gamma(0)) = (0, \sqrt{1 + |\mathbf{L}|^2})$ and $\hat{\sigma}(\gamma(t)) = (0, \sqrt{|\mathbf{K}|^2 + H^2})$. Note that $A(z,t) \to \infty$ implies the existence of a $T > 0$ so that $t > T$ implies $\hat{\sigma}(\gamma(t)) > \hat{\sigma}(\gamma(0))$. Henceforth assume that $t > T$.

Letting $B_0 = d\big(\gamma(0), \hat{\sigma}(\gamma(0))\big) = d\big(\gamma(0), \Gamma_{\tilde{z}}\big)$ and Δ is as above,

$$\begin{aligned}
A &= d\big(\hat{\sigma}(\gamma(0)), \hat{\sigma}(\gamma(t))\big)\\
B &= \Delta(\gamma(t)) - B_0\\
D &= d\big(\gamma(0), \gamma(t)\big).
\end{aligned}$$

Using the distance formulas (1.1),

$$\begin{aligned}
\cosh(\Delta) &= \frac{\sqrt{|\mathbf{K}|^2 + H^2}}{H}\\
e^A &= \sqrt{\frac{|\mathbf{K}|^2 + H^2}{|\mathbf{L}|^2 + 1}}.
\end{aligned}$$

Solving yields

$$\begin{aligned}
H &= \frac{e^A}{\cosh(\Delta)}\sqrt{1 + |\mathbf{L}|^2}\\
|\mathbf{K}| &= e^A \tanh(\Delta)\sqrt{1 + |\mathbf{L}|^2}.
\end{aligned}$$

From (2.3), $\frac{1}{H}\frac{d|\mathbf{K}|}{dt}$ is bounded and so computing, $\operatorname{sech}(\Delta)\dot{\Delta} + \sinh(\Delta)\dot{A}$ is bounded, and so

$$\dot{A} < \frac{k}{\sinh(\Delta)} - \frac{\dot{\Delta}}{\sinh(\Delta)\cosh(\Delta)} \tag{2.4}$$

for some constant k.

Now using (1.6) and (1.7) we may write $A = c_1 t + o(t)$, $B = c_2 t + o(t)$ and $D = c_3|t| + o(t)$ with $c_3 > 0$ by assumption. If $c_2 \neq 0$, since $\Delta(\gamma(t)) = B(z,t) - B_0$, (2.4) yields that

$$\int_0^\infty \dot{A} < \infty$$

contradicting the fact that $A(z,t) \to \infty$, and thus $c_2 = 0$. Finally note that the distance formulas (1.1) also yield that

$$\cosh(D) = 1 + \frac{|\mathbf{K} - \mathbf{L}|^2 + (H-1)^2}{2H}$$

and so $c_1 = c_3$ as required. $\square$

Remark 2.4: Given $\lambda > 1$ and $\epsilon > 0$, a curve $\gamma : \mathbb{R} \to \mathbb{P}$ is called a (λ, ϵ)-quasigeodesic if

$$\lambda^{-1}(d-c) - \epsilon \leq d(\gamma(c), \gamma(d)) \leq \lambda(d-c) + \epsilon$$

for all $[c,d]$ in the domain of γ. A quasigeodesic always has a shadowing geodesic Γ and there is a constant k depending only on λ and ϵ so that γ is within a distance k of Γ (see [GH] or [CDP]). This implies that near S_∞ a quasigeodesic lies in a cone with vertex at its limit point. For a generic point of an ergodic positive progress measure Lemma 2.2

ensures the existence of shadowing geodesic for the trajectory through the point. However, in general, the trajectory will not be a quasigeodesic, and the distance from the shadowing geodesic may become unbounded while still being $o(t)$ as required by Lemma 2.3. In the example of §5 this happens for almost every point for an ergodic measure. Rather than being contained in a cone based on S_∞, the envelope of the trajectories are tangent to the sphere at infinity.

Section 3: The rotation measure.

§**3.1 Definition of the rotation measure.** We give two equivalent definitions of the rotation measure, one in the base and the other in the cover. Fix $z \in \mathcal{B}$ and $t \in \mathbb{R}$ and let G be the oriented geodesic segment that is homotopic with fixed endpoints to the projection of the orbit segment starting at z and flowing for time t, *i.e.* to $p(\phi_{[0,t]}(z))$. The lift of G to $T_1 M$ is denoted G' and $M(z,t)$ is the uniformly distributed measure supported on G' that has total mass equal to the hyperbolic length of G. Note that this length is $D(z,t)$, the distance cocycle. The *rotation measure* is

$$\rho(x) = \lim_{t\to\infty} \frac{M(z,t)}{t} \tag{3.1}$$

if the weak limit exists.

For the second definition, fix a lift $\tilde{z}$ to $\tilde{\mathcal{B}}$ of z and let $\tilde{G} = [p(\tilde{z}), p(\tilde{\phi}_t(\tilde{z}))]$ and $\tilde{G}'$ its lift to $T_1\tilde{M}$. Now let $M(z,t) = \pi_*(\mu)$ where μ is the arc length measure on $\tilde{G}$ lifted to $\tilde{G}'$. The rotation measure is once again defined as in (3.1).

Note that if the rotation measure exists it is an invariant measure for the geodesic flow g_t. It is in general not a probability measure, but rather has total mass equal to $D^*(z)$.

§**3.1 Existence of the rotation measure.** If $M(z,t)$ were an measure-valued, additive cocycle, then Lemma 1.3 would immediately give the existence of the rotation measure almost everywhere. However, it does not have the appropriate additive properties, but it is asymptotic to the measured-valued cocycle defined as follows.

Given $z \in Z_2$ with Z_2 as in the proof of Lemma 2.3 and $t \in \mathbb{R}$, let $N(z,t)$ be the signed uniformly distributed measure on the segment of the geodesic flow $g_{[0,A(z,t)]}(\sigma(\tilde{z}))$ (this orbit segment has as its endpoints $\sigma(\tilde{z})$ and $\sigma(\tilde{\phi}_t(\tilde{z}))$). The total mass of $N(z,t)$ is thus $A(z,t)$. Now clearly $N(z,t)$ is a measured valued cocycle for ϕ_t and $N_{\mathbf{1}}(z,1) = A(z,1)$. Thus if η has compact support, $N_{\mathbf{1}}(z,1) \in L^1(\eta)$ and so using Lemma 1.3, the weak limit $N^*(z)$ exists on a full measure set denoted Z_N. The next lemma shows that these limits are the same as those for the rotation measure ρ. In particular, even though $N(z,t)$ is a signed measure, the limit N^* is a ordinary (i.e. non-negative valued) measure.

Proposition 3.1 *Given ergodic η as in Lemma 2.3, then for almost every z, $\rho(z) = N^*(z)$.*

Proof: Fix a $z \in Z_3 := Z_2 \cap Z_N$ where Z_2 is as in Lemma 2.3 and Z_N as above. We shall show that

$$\frac{N(z,t)}{t} - \frac{M(z,t)}{t} \to 0$$

weakly. It suffices (see Theorem 7.1 in [Bi]) to show that for any C^1 function $f : T_1M \to \mathbb{R}$,

$$\left|\frac{\int f\ dN(z,t)}{t} - \frac{\int f\ dM(z,t)}{t}\right| \to 0. \tag{3.2}$$

Formula (3.2) is proved by working in the covering space $\mathbb{H}$. Let f also denote the lift of f to $T_1\tilde{M} \to \mathbb{R}$ and fix a lift $\tilde{z}$, and $t > 0$ large enough to satisfy conditions given below. Let $\gamma(t) = p(\tilde{\phi}_t(\tilde{z}))$, $\Gamma = \Gamma_{\tilde{z}}$, and β be a parameterization of $[\gamma(0), \gamma(t)]$ by arclength. Let A, B and D be the cocycles defined in §2 and drop the dependence of cocycles, etc. on z and t. With this convention, $\beta : [0, D] \to \mathbb{H}$ with $\beta(0) = \gamma(0)$, $\beta(D) = \gamma(t)$, and $\|\dot{\beta}\|_h \equiv 1$. Also, let α be a parameterization of $[\sigma(\gamma(0)), \sigma(\gamma(t))] \subset \Gamma$ by arclength, and so $\alpha : [0, A] \to \mathbb{U}$ has $\alpha(0) = \sigma(\gamma(0))$, $\alpha(A) = \sigma(\gamma(t))$, and $\|\dot{\alpha}\|_h \equiv 1$. Let s and u be the parameters of α and β respectively. Define the length of the geodesic segment $[\sigma(\gamma(0)), \gamma(t)]$ to be $L(t)$ and let $\delta : [0, L] \to [\sigma(\gamma(0)), \gamma(t)]$ be parameterization by arclength.

Since η is positive progress, Lemma 2.3 yields $A^*(\eta) = D^*(\eta) > 0$ and $B^*(\eta) = 0$. Thus for all sufficiently large t, $A(t), L(t) > B(t)$, and $L(t), D(t) > d(\gamma(0), \sigma(\gamma(0)) := B_0$, and $A(t), L(t), D(t) > 1$ also. Thus expressing the left hand side of (3.2) in coordinates in the cover

$$\begin{aligned}
&\frac{1}{t}\left|\int_0^A f(\alpha(s))\ ds - \int_0^D f(\beta(u))\ du\right| \\
&\leq \frac{1}{t}\left|\int_0^A f(\alpha(s)) - \int_0^L f(\delta(v))\ dv\right| + \frac{1}{t}\left|\int_0^L f(\delta(v))\ dv - \int_0^D f(\beta(u))\ du\right| \\
&\leq \frac{1}{t} m(5(B + B_0) + 2)
\end{aligned}$$

Where the last inequality uses Lemma 1.1. The constant m as defined in that lemma is finite because f is the lift of a C^1-function on the compact manifold T_1M. Thus since $B = o(t)$, (3.2) follows. □

Proposition 3.1 immediately yields the main existence theorem for the rotation measure.

Theorem 3.2: *If ϕ_t is a C^1-flow on $\mathcal{B}$ and η is an ergodic, ϕ_t-invariant probability measure with compact support and $D^*(\eta) > 0$, then the rotation measure $\rho(z)$ exists almost everywhere and has a constant value denoted $\rho(\eta)$.*

Section 4: Properties of the rotation measure.

§**4.1 Semiconjugacy and the rotation measure.** The rotation measure $\rho(\eta)$ can be given additional dynamical meaning by connecting its behavior as an invariant measure of the hyperbolic geodesic flow with the dynamics of the ϕ_t-invariant measure η. This connection is provided by a measure theoretic semi-conjugacy induced by the projection σ from a generic orbit to its shadowing geodesic. Although σ takes ϕ_t-orbits to g_t-orbits, it is perhaps not locally injective on orbits; it thus does not give a semiconjugacy. Informally, the difficulty is that ϕ_t-orbits are only asymptotically in the same direction as the shadowing geodesic; they certainly can travel forward and backwards in the direction of the geodesic . In more precise language, the cocycle $A(z,t)$ which measures the *signed* length of the projection of the orbit onto the shadowing geodesic can certainly be negative for some t. However, $A^*(\eta) = D^*(\eta) > 0$ for positive progress measures η by Lemma 2.3, and so Lemma 1.4 provides a monotone cocycle that "straightens out" σ into a semiconjugacy.

Two measures are said to be equivalent if they are mutually absolutely continuous; this is denoted $\eta_1 \sim \eta_2$. If η is an invariant measure for the flow ϕ_t, the $h_\eta(\phi_t)$ denotes its metric entropy.

Theorem 4.1: *Let ϕ_t be a C^1-flow on $\mathcal{B}$, η an ergodic, ϕ_t-invariant probability measure with compact support and $D^*(\eta) > 0$. The geodesic flow g_t on T_1M is induced by a hyperbolic metric and the projection $\sigma : Z_3 \to T_1M$ is defined in §2.3 with Z_3 the full measure set defined in Proposition 3.1. If μ is the measure $\rho(\eta)/D^*(\eta)$ defined on T_1M then*
(a) $(\mathcal{B}, \phi_t, \eta)$ is semi-conjugate to (T_1M, g_t, μ),
(b) μ is the unique, ergodic, g_t-invariant probability measure equivalent to $\sigma_*(\eta)$,
(c) $h_\eta(\phi_t) \geq h_\mu(g_t)/D^*(\eta)$.

Proof: A semiconjugacy result is proved first. Recall the Lipschitz cocycle A defined above Lemma 2.3. Lemma 2.3 says that $A^*(\eta) = D^*(\eta)$, and the latter is positive by hypothesis. Thus by Lemma 1.4 there is an invertible Lipschitz cocycle $\hat{A}$ with $A^*(\eta) = \hat{A}^*(\eta)$ with $\hat{A}$ defined almost everywhere by

$$\hat{A}(z,t) = A(z,t) + \beta(\phi_t(z)) - \beta(z)$$

for a measurable, real valued β that is continuous on orbits.

Define $\hat{\sigma} : Z_3 \to T_1M$ as $\hat{\sigma}(z) = g_{\beta(z)}(\sigma(z)) = \sigma(z) + \beta(z)$ where in the last formula we add on an oriented geodesic using a parameterization by arc length. The definition of A yields that $\sigma \circ \phi_t(z) = g_{A(z,t)} \circ \sigma(z)$, and so $\hat{\sigma} \circ \phi_t(z) = g_{\hat{A}(z,t)} \circ \hat{\sigma}(z)$, for all z and t

Since $\hat{A}$ is monotone, as described in §1.5, it may be used to define a time-changed flow $\hat{\phi}_s$ with $\hat{\sigma}\hat{\phi}_s(z) = g_s\hat{\sigma}(z)$, for all z, s. There is a unique ergodic $\hat{\phi}_s$-invariant probability measure $\hat{\eta}$ that is equivalent to η. Thus $\hat{\sigma}_*(\hat{\eta})$ is a g_t-invariant, Borel probability measure that is ergodic since $\hat{\eta}$ is. Now define $\hat{\mu} = \hat{\sigma}_*(\hat{\eta})$, and note that (a) has been proven with $\hat{\mu}$ in place of μ. To prove (b) with the same replacement we must show that $\hat{\sigma}_*(\hat{\eta}) \sim \sigma_*(\eta)$ (uniqueness follows from ergodicity).

Since η is ϕ_t-invariant, a set V has $\eta(V) = 0$ if and only if its saturation $\mathbb{R} \cdot V := \{\phi_t(x) : t \in \mathbb{R}, x \in V\}$ has measure zero. Now for every z, $\sigma(z)$ and $\hat{\sigma}(z)$ lie on the same ϕ_t trajectory, and thus $\mathbb{R} \cdot \sigma^{-1}(U) = \mathbb{R} \cdot \hat{\sigma}^{-1}(U)$ for any Borel $U \subset T_1(M)$. Thus $\hat{\sigma}_*(\eta) \sim \sigma_*(\eta)$. But $\eta \sim \hat{\eta}$ since they correspond under a time change, thus $\hat{\mu} = \hat{\sigma}_*(\hat{\eta}) \sim \hat{\sigma}_*(\eta) \sim \sigma_*(\eta)$.

To complete the proof of (a) and (b) we must show that $\hat{\mu} = \rho(\eta)/D^*(\eta)$. It is standard that since $\hat{\mu}$ is ergodic for g_t, there is a full $\hat{\mu}$-measure set $X \subset T_1M$ so that $x \in X$ implies that the unit mass measures distributed uniformly with respect to time on the orbit segments $g_{[0,s]}(x)$ converge weakly to $\hat{\mu}$ as $s \to \infty$. Thus since $\hat{\mu} = \hat{\sigma}(\hat{\eta})$, the full $\hat{\eta}$-measure set $\hat{\sigma}^{-1}(X)$ is such that $z \in \hat{\sigma}^{-1}(X)$ implies that $\hat{\sigma}(z) = x$ has this property. Thus for such a z, if $Q(z,s)$ is the measure on the geodesic segment $[\hat{\sigma}(z), \hat{\sigma}(\hat{\phi}_s(z))] = [\hat{\sigma}(z), g_s(\hat{\sigma}(z))]$ that is uniformly distributed with respect to arc length, then

$$\frac{Q(z,s)}{s} \to \hat{\mu}$$

weakly.

Now let $z \in \hat{\sigma}^{-1}(X) \cap Z_3$. By construction, $Q(z, s(z,t)) - N(z,t)$, with N as in §3, has total mass between 0 and $\beta(\phi_t(z)) - \beta(z)$. Lemma 1.4 says that $\hat{A}^*(\eta) = A^*(\eta)$ and so (1.8) implies that $\beta(\phi_t(z))/t \to 0$ as $t \to \infty$. Thus weakly,

$$\frac{Q(z, s(z,t))}{t} \to N^*(z) = \rho(\eta)$$

using Proposition 3.1. But also

$$\frac{s(z,t)}{t} = \frac{\hat{A}(z,t)}{t} \to \hat{A}^*(\eta) = A^*(\eta) = D^*(\eta)$$

by Lemmas 1.4 and 2.3. Thus

$$\lim_{t\to\infty} \frac{Q(z, s(z,t))}{t} = \lim_{t\to\infty} \frac{s(z,t)}{t} \; \frac{Q(z, s(z,t))}{s(z,t)} = D^*(\eta)\, \hat{\mu},$$

proving that $\hat{\mu} = \rho(\eta)/D^*(\eta)$. The entropy statement in (c) follows from the formula for entropy change of a flow under a time change. □

Remark 4.1: If X is a compact invariant set consisting of quasigeodesics with the same constants, then an alteration of the proof of Theorem 4.1 yields a continuous semi-conjugacy onto a compact invariant subset of the geodesic flow. This construction is done in [BG] for the case of Euler-Lagrange systems whose configuration space is a hyperbolic manifold (eg. the geodesic flows of a Riemannian metric).

§4.2 The rotation measure as a function. To view the rotation measure ρ as a function its domain and codomain must be specified. If for a continuous flow ϕ_t on a metric space X, $\mathcal{E}(\phi_t, X)$ denotes the space of ergodic, ϕ_t-invariant probability measures with compact support, then the domain of ρ consists of the positive progress measures in $\mathcal{E}(\phi_t, \mathcal{B})$. Theorem 4.1 says that the codomain of ρ consists of the ergodic measures of the geodesic flow. It is usually more convenient to work with probability measures, so $\rho(\eta)$ is considered a pair consisting of "direction" and "magnitude" with the direction given by an invariant probability measure of the geodesic flow. Accordingly (with an alteration of notation to avoid confusion) let

$$\hat{\rho}(\eta) = (\frac{\rho(\eta)}{D^*(\eta)}, D^*(\eta)) \in \mathcal{E}(g_t, T_1 M) \times \mathbb{R}.$$

To include the case of zero progress measures, define $\hat{\rho}(\eta) = 0$ if $D^*(\eta) = 0$. Thus if

$$\hat{\mathcal{E}}(g_t, T_1 M) := (\mathcal{E}(g_t, T_1 M) \times \mathbb{R}) \cup \{0\},$$

then $\hat{\rho} : \mathcal{E}(\phi_t, \mathcal{B}) \to \hat{\mathcal{E}}(g_t, T_1 M)$.

The map $\hat{\rho}$ can be extended to the set of all ϕ_t-invariant measures using the ergodic decomposition. This must be approached with caution if one of the measures in the decomposition of the measure m has zero progress. In this case it is no longer guaranteed that the generic point for the measure m has a shadowing geodesic despite the fact that the extended function $\hat{\rho}(m)$ has nonzero value.

Also note that the rotation measure is not invariant under topological conjugacy of the flow. This is because the average speed can be different for a pair of measures corresponding under the conjugacy. However, the first component of the image of $\hat{\rho}$ is conjugacy invariant, and the necessary adjustment to the second component is easily computed using the time change cocycle.

§**4.3 The homology rotation vector.** In this section we show that the rotation measure is a generalization of the homology rotation vector in the sense of (4.1) below. The homology rotation vector goes back to Schwartzman ([Sc]) who called it an asymptotic cycle, because it gives a dynamical meaning to elements of real homology with "irrational slope". He defined the rotation vector for individual points, and pointed out that Birkhoff's Ergodic Theorem implies that the generic points for an ergodic measure all have the same rotation vector. This allows the assignment of a real homology class to an ergodic invariant measure. If one considers the flow and invariant measure as an oriented lamination with a transverse measure, then this homology class corresponds to the geometric current of the measured lamination ([RS]).

We recall the definition of the homology rotation vector in the context used here. Pick a de Rham basis for $H^1(M;\mathbb{R}) \simeq \mathbb{R}^\beta$, where β is the first Betti number of M, and let the closed one-forms $\lambda_1, \lambda_2, \ \ldots \ , \lambda_\beta$ represent the basis elements. Given $z \in \mathcal{B}$ and $t \in \mathbb{R}$, define $S : \mathcal{B} \times \mathbb{R} \to \mathbb{R}^\beta$ component-wise as

$$(S(x,t))_i = \int_{p(\phi_{[0,t]}(z))} \lambda_i.$$

The definition depends on the choice of basis for $H^1(M;\mathbb{R})$, but not on the one-forms representing the chosen basis. It is clear that S is a vector-space-valued, additive cocycle. If $S^*(z) \in H_1(M;\mathbb{R}) \simeq \mathbb{R}^\beta$ exists it is called the *homology rotation vector* of z under the flow ϕ_t. Roughly speaking, the direction of the vector $S^*(z)$ is the direction of the motion of the orbit around the manifold as given in homology, and the magnitude of the class is the speed of the orbit (or more properly, the rate of progress in the universal free Abelian cover, cf. Remark 2.1 about ℓ^* vs. D^*). If η is an ergodic invariant measure with compact support, then by Theorem 1.2, S^* exists almost everywhere and has constant value denoted $S^*(\eta)$. Thus as a function, $S^* : \mathcal{E}(\phi_t, \mathcal{B}) \to \mathbb{R}^\beta$.

If μ is a ergodic g_t-invariant probability measure on T_1M, we may make the analogous definition of a homology rotation vector under the geodesic flow. To prevent confusion the $\mathbb{R}^\beta$-valued cocycle in this case is called T instead of S, thus $T^* : \mathcal{E}(g_t, T_1M) \to \mathbb{R}^\beta$. This maybe extended in the natural way to a function defined on the range of $\hat{\rho}$ as
$\hat{T} : \hat{\mathcal{E}}(g_t, T_1M) \to \mathbb{R}^\beta$ defined by $\hat{T}(\mu, r) = rT^*(\mu)$ and $T(0) = 0$.

The precise manner in which the rotation measure extends the homology rotation vector is expressed by

$$\hat{T} \circ \hat{\rho} = S^*. \tag{4.1}$$

To prove this note that the closed one-forms λ_i may be treated as maps $T_*M \to \mathbb{R}$ that are linear on fibers. These maps restrict to continuous functions on T_1M. Thus for $\mu \in \mathcal{E}(g_t, T_1M)$ the Birkhoff ergodic theorem yields that

$$(T^*(\mu))_i = \int \lambda_i \, d\mu. \tag{4.2}$$

Since the one-forms are closed, the integral of λ_i over the curve segment $p(\phi_{[0,t]}(z))$ is the same as the integral of λ_i over the geodesic segment that is homotopic to this curve segment rel endpoints. Recall that the measure $M(z,t)$ defined in §3 is uniformly distributed with respect to arc length on this curve segment segment. Thus if z is generic for S and M and interpreting the one-forms as $\lambda_i : T_1M \to \mathbb{R}$ in the second integral,

$$(S^*(\eta))_i = \lim_{t\to\infty} \frac{1}{t} \int_{p(\phi_{[0,t]}(z))} \lambda_i = \lim_{t\to\infty} \int \lambda_i \frac{dM(z,t)}{t} = \int \lambda_i \, d\rho(\eta) \tag{4.3}$$

using Proposition 3.1.

For $\eta \in \mathcal{E}(\phi_t, \mathcal{B})$, the definition of $\hat{\rho}$ and (4.2) yields

$$(\hat{T}(\hat{\rho}(\eta))_i = (D^*(\eta) \, \frac{T^*(\rho(\eta))}{D^*(\eta)})_i = \int \lambda_i \, d\rho(\eta).$$

By (4.3) this is $(S^*(\eta))_i$, proving (4.1).

Section 5: Examples and Applications.

This section gives an example that illustrates various results and definitions of the previous sections. The justifications of many statements are left for the reader, and knowledge of basic definitions and constructions from dynamical systems theory is assumed. For an introduction to this material see [HK], [Fk] or [Ro]. The analysis of the example makes use of various results about Cayley graphs of the fundamental groups of hyperbolic surfaces embedded as equivariant trees in $\mathbb{P}^2$, random walks on these trees, subshifts of finite type, and the symbolic coding of geodesics. Most of this material is surveyed in [BKS] (see also [S]). For an introduction to random walks on trees see [PL] or [Wo]. Basic results about symbolic dynamics are well covered in [Kt].

The main example is a diffeomorphism Φ of T, the two-dimensional torus minus an open disk. Note that T is a hyperbolic manifold. It is not closed but it can be embedded in a closed genus two surface, and Φ extended as the identity outside the embedded copy of T. We focus mainly on the dynamics of Φ on T, bringing the ambient surface into play only when necessary. The relevant dynamics of Φ lie in a compact, invariant, transitive hyperbolic set Ω. The Markov partition for Ω has rectangles that are labeled with generators of $\pi_1(T)$ (as in [W]) which describe the motion of the box under one iterate. Thus the symbolic description of an orbit describes the motion of the orbit around T, or equivalently, in the universal cover $\tilde{T}$.

The symbolic description can also be used to generate a walk on the Cayley graph of $\pi_1(T) \simeq F(a,b)$, the free group on two generators. When this graph is embedded as an equivariant spine $\mathcal{T}$ of $\tilde{T}$, the intrinsic geometry of the graph is closely related to the

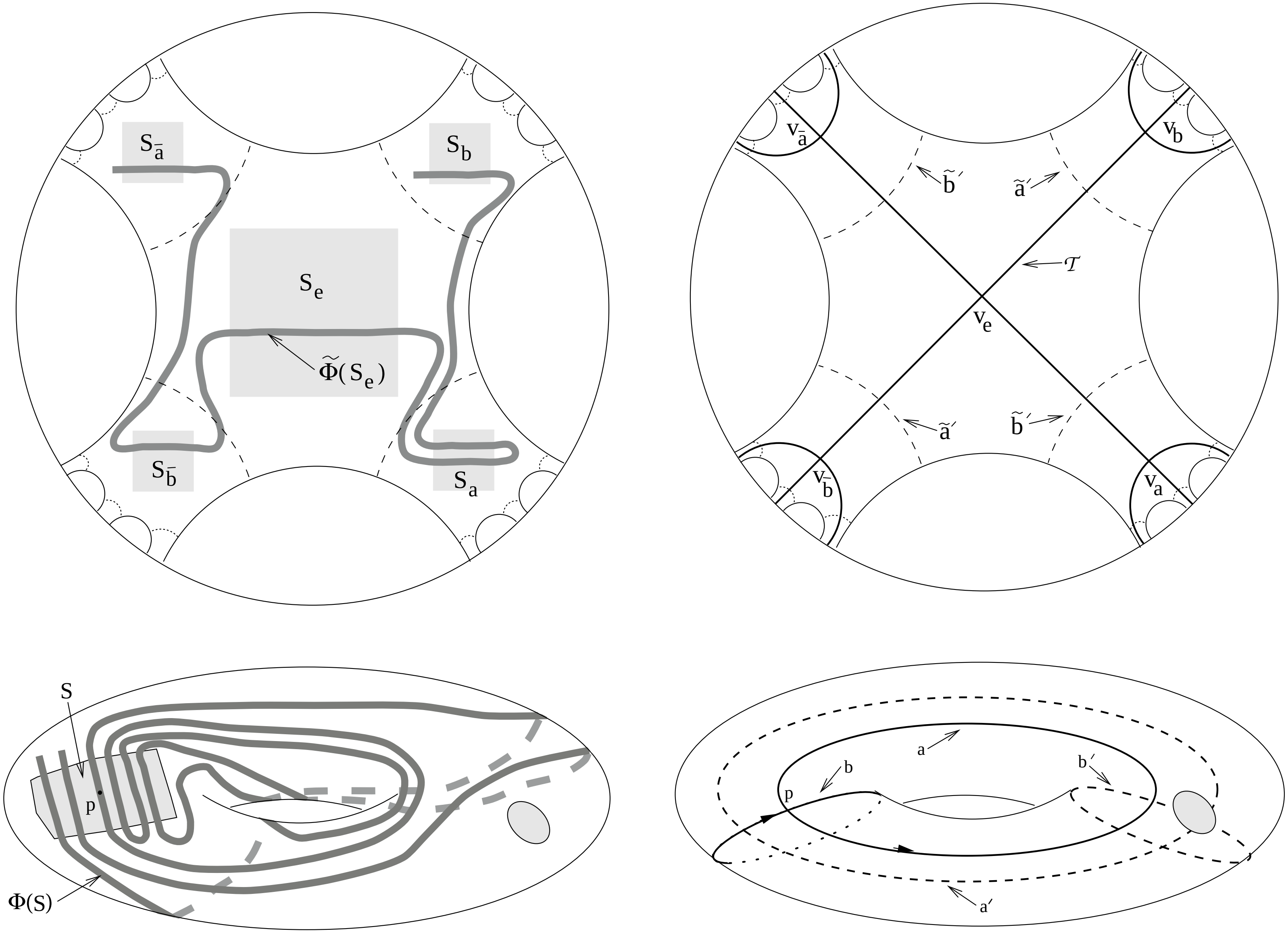

Figure 1: (counterclockwise from the upper left): (a) One lift of (b). (b) The image of the rectangle S under the map Φ. (c) The projection of (d). (d) The tree $\mathcal{T}$ in the universal cover.

ambient hyperbolic geometry, and walks on the graph shadow the lifted orbit of Φ with the same symbolic coding. Thus the relationship of the dynamics of Φ on Ω to the ambient hyperbolic geometry can be analyzed using the symbolic description. In particular, symbolic analogs of the various geometric functions and cocycles of the previous sections can be defined and computed within the symbolic framework.

Although this section focuses on a single example, a similar (but more complicated) analysis can be given for a much wider class of maps, for example, for Axiom A diffeomorphisms isotopic to the identity on a hyperbolic manifold.

§5.1 The diffeomorphism and its Markov partition. The construction is a version of the standard one for Axiom A diffeomorphisms. Let S be a square embedded in the interior of T. The diffeomorphism $\Phi : T \to T$ is isotopic to the identity and stretches S and lays it over itself in a collection of linear strips. We focus attention on five of these with each consisting of points that have traversed T along the same element of the fundamental group, these elements being the generators a and b, their inverses $\bar{a}$ and $\bar{b}$, and the identity element e. Figure 5.1(a) shows the image of S in T. Figure 5.1(b) shows the image of one lift of S under the lift of Φ to the universal cover.

The invariant set of interest is defined by

$$\Omega = \bigcap_{n \in \mathbb{Z}} \Phi^n(S).$$

Note that Ω is an invariant Cantor set but it is not isolated in the nonwandering set, and so it is not a basic set. However its dynamics can be described symbolically as follows. Fix a hyperbolic metric on T with geodesic boundary such that the geodesics in the free homotopy classes of a and b both have length one. Let $\tilde{T} \subset \mathbb{P}^2$ be a geometric universal cover of T, i.e. $\tilde{T}$ modulo covering transformations is isometric to T with the chosen hyperbolic metric. The covering transformation that corresponds to an element $w \in \pi_1(T)$ is denoted τ_w. Thus the deck group is generated by a pair of isometries of $\mathbb{P}^2$, τ_a and τ_b. Note that elements in $\pi_1(T)$ are written left to right, while composition of deck transformations are written in the other order. Thus, for example, $\tau_{ab} = \tau_b \circ \tau_a$. Treating this representation of $\pi_1(T)$ as a Fuchsian group leads to the name *limit set* for the portion of the topological frontier of $\tilde{T}$ that lies in S_∞.

Fix a fundamental domain in $\tilde{T}$ and an injective lift of S contained in that fundamental domain. Denote this distinguished upstairs copy of S as S_e. The other lifted copies of S are labeled by the deck transformation that takes S_e to them, so for each $w \in \pi_1(T)$, $S_w := \tau_w(S)$ (see Figure 5.1b). The lifted copy of Ω contained in S_w is denoted Ω_w. Let $\tilde{\Phi}$ be the lift of Φ that extends to the identity on the limit set in S_∞. The Markov rectangles are first defined in the cover by choosing $s \in \{a, \bar{a}, b, \bar{b}, e\}$ and letting $\tilde{R}_s$ denote the set of points in S_e that are mapped into S_s. Thus $\tilde{R}_s = \tilde{\Phi}^{-1}(S_s) \cap S_e$. The Markov rectangles $R_s \subset S$ are the projections of the $\tilde{R}_s$.

With this Markov partition the symbolic model of Φ restricted to Ω is the collection Σ of all two-sided sequences with elements taken from the set $\{a, \bar{a}, b, \bar{b}, e\}$, $\Sigma = \{a, \bar{a}, b, \bar{b}, e\}^{\mathbb{Z}}$. Define $\iota : \Omega \to \Sigma$ as the itinerary map with respect to the partition, i.e. s is the i^{th} symbol in the sequence $\iota(x)$ exactly when $\Phi^i(x)$ is in the subrectangle R_s. The map ι is a homeomorphism that conjugates Φ restricted to Ω to the shift map δ on Σ; $\delta \circ \iota = \iota \circ \Phi_{|\Omega}$.

(The shift on a symbol space is denoted δ here rather than the usual σ to avoid confusion with the projection σ defined in §2.3.)

The labeling of the Markov rectangles was chosen so that the symbolic description of an orbit would describe the motion of its lift in the cover. The first i-steps of this motion are described by the first i-letters in the symbol sequence, which is considered as an word in $\pi_1(T)$. Accordingly, for a sequence $\mathbf{s} \in \Sigma$, and a non-zero integer i, $w(\mathbf{s}, i) \in \pi_1(F)$ is defined by

$$w(\mathbf{s}, i) = s_0 s_1 \dots s_{i-1} \text{ if } i > 0$$
$$w(\mathbf{s}, i) = \bar{s}_{-1} \bar{s}_{-2} \dots \bar{s}_i \text{ if } i < 0$$

where an overbar denotes the inverse, and let $w(\mathbf{s}, 0) = e$. If $x \in \Omega$ is coded by the sequence $\mathbf{s} = \iota(x)$ and $\tilde{x} \in \Sigma_e$ is its lift, then $\tilde{\Phi}^i(\tilde{x}) \in S_{w(\mathbf{s},i)}$, for all i.

This description makes it clear that each lifted orbit from Ω is behaving like a discrete walk in which each step consists of a jump into one of the four adjacent fundamental domains, or else a pause step in which the orbit stays in the same copy of S. Other types of walks correspond to various invariant subsets of Ω. Using the conjugacy ι these subsets can be defined in the symbol space Σ. The standard walks without pauses correspond to sequences which do not contain the symbol e. The collection of these is $\Sigma' := \{a, \bar{a}, b, \bar{b}\}^{\mathbb{Z}} \subset \Sigma$, and the corresponding subset of Ω is $\Omega' = \iota^{-1}(\Sigma')$. Walks without backtracking, also called self-avoiding walks, correspond to sequences from Σ' where the transitions $a \to \bar{a}$, $\bar{a} \to a$, $b \to \bar{b}$, and $\bar{b} \to b$ are *not* allowed. The collection of these sequences is denoted $\hat{\Sigma}$ and is a subshift of finite type with transition matrix

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

Let $\hat{\Omega} = \iota^{-1}(\hat{\Sigma})$. Note that the allowable finite blocks of symbols for $\hat{\Sigma}$ are the same as the reduced words in $F(a, b)$.

§5.2 The equivariant tree, walks and symbolic coding. This subsection formalizes the connection between the dynamics of $\tilde{\Phi}$, walks, and symbolic coding using a tree $\mathcal{T} \subset \tilde{T}$ defined as follows. Let p be the fixed point of Φ that is coded by e^∞. Adjust the position of S and p so that p lies at the intersection of the two closed geodesics in T that represent the free homotopy classes of the two generators a and b, respectively. The tree $\mathcal{T}$ is defined as the total lift of this pair of closed geodesics. It is an infinite simplicial tree with four edges coming into each vertex. For each $w \in F(a, b)$ the vertex contained in S_w is denoted v_w. The edges of $\mathcal{T}$ connect just those vertices in adjacent fundamental domains and are geodesic segments with length one. The tree $\mathcal{T}$ can also be identified with the Cayley graph of $\pi_1(M) \simeq F(a, b)$ (see Figure 5.2b).

If we define the distance $d_{\mathcal{T}}(v_{w_1}, v_{w_2})$ between vertices in $\mathcal{T}$ as the hyperbolic length of the unique topological arc embedded in $\mathcal{T}$ that connects v_{w_1} and v_{w_2}, then this distance is the same as the distance between the two elements w_1 and w_2 in the word metric on $F(a, b)$, namely the length of the reduced word $\bar{w}_1 w_2$. In symbols, $d_{\mathcal{T}}(v_{w_1}, v_{w_2}) = \ell(\bar{w}_1 w_2)$, where for $w \in F(a, b)$, $\ell(w)$ denotes the reduced length of w, i.e. the number of generators in the shortest word representing w. A classic result of Milnor [Mi] says that this distance in $\mathcal{T}$ is equivalent to the hyperbolic distance in $\tilde{T}$ in the sense that,

$$c_1 d_{\mathcal{T}}(v_{w_1}, v_{w_2}) \leq d(v_{w_1}, v_{w_2}) \leq c_2 d_{\mathcal{T}}(v_{w_1}, v_{w_2}) \tag{5.1}$$

for some positive constants c_1 and c_2.

To connect walks on $\mathcal{T}$ to the dynamics of Φ we shall need to consider two-sided (i.e. indexed by $\mathbb{Z}$) walks because Φ is invertible. It is also convenient to just consider standard walks without pauses; these correspond to sequences from Σ'. The focus in this subsection is on individual walks; measures on the collection of walks will be considered in the last subsection. A ***walk*** on $\mathcal{T}$ is an bi-infinite sequence of adjacent vertices $\dots, v_{w_{-2}}, v_{w_{-1}}, v_{w_0}, v_{w_1}, v_{w_2}, \dots$. The vertex v_{w_0} (or sometimes w_0) is called the ***root*** of the walk. The direction of each step is given by $\bar{w}_i w_{i+1}$ which is an element of the set $\{a, \bar{a}, b, \bar{b}\}$. Thus the walk is alternatively specified by the root w_0 and the sequence $\mathbf{s} \in \Sigma'$ with $s_i = \bar{w}_i w_{i+1}$. If a walk with root w_0 and the point $\tilde{x} \in S_{w_0}$ have the same symbolic description from Σ', then at each step (iterate of $\tilde{\Phi}$) they are at most the diameter of S apart. In dynamical language, this says that the walk with root w_0 described by the symbol sequence $\mathbf{s}$ is a pseudo-orbit for $\tilde{\Phi}$ which is shadowed by the orbit of the lift $\tilde{x} \in S_{w_0}$ of the point $x \in \Omega$ with $\iota(x) = \mathbf{s}$.

The next step is to identify the symbolic analog of the geodesic flow using walks on $\mathcal{T}$. A walk with sequence $\mathbf{t} \in \hat{\Sigma}$ is called a ***geodesic walk*** because it has no backtracking and always converges to an α and ω limit points in S_∞. Since a geodesic walk is clearly contained in a quasigeodesic (see Remark 2.4), the walk is a uniformly bounded distance away from the unique hyperbolic geodesic in $\mathbb{P}^2$ which has the same limit points on S_∞. Further, this bound is the same for all geodesic walks. A point on a symbolic geodesic is specified by the root of a geodesic walk. Moving the root to an adjacent vertex on the walk requires a shift of the specifying sequence $\mathbf{t}$. Thus the discrete dynamical system $(\hat{\Sigma}, \delta)$ is the symbolic analog of the geodesic flow on T. Note that elements of $\hat{\Sigma}$ correspond to points on geodesics, the geodesic itself is represented by the *orbits* of points. In algebraic language, a bi-infinite word using the symbols $a, \bar{a}, b, \bar{b}$ uniquely specifies a geodesic in T, but the specification of a point on this geodesic requires an explicit numbering of elements, i.e. the insertion of a "decimal point".

A topological conjugacy between the geodesic flow and the suspension flow of $(\hat{\Sigma}, \delta)$ makes the correspondence more precise. Since T has boundary, the meaning of "geodesic flow" must be clarified. Consider T as embedded in a closed genus two surface M with a hyperbolic metric that restricts to the chosen one on T. Let X consist of all the geodesics of M that are wholly contained in T including the boundary geodesic with both orientations. Then X as a subset of $T_1 M$ is compact and invariant under the geodesic flow g_t.

The definition of the conjugacy requires a more careful choice of fundamental domain for $\tilde{T}$. Let T_e be a fundamental domain whose boundary consists of four pieces which are lifts of pieces of the boundary of T and four pieces (called edges) that are lifts of the geodesic arcs labeled a' and b' in Figure 5.2a, and let $T_w = \tau_w(T_e)$. This choice of fundamental domain ensures that for any $\Gamma \in \tilde{X}$ and $w \in \pi_1(T)$, $\Gamma \cap T_w$ is either empty or else a single closed interval joining distinct edges of T_w.

Given a copy T_w of the fundamental domain and a geodesic $\Gamma \in \tilde{X}$ that intersects it nontrivially, there is a unique geodesic walk rooted at w that the geodesic shadows. If this walk has sequence $\mathbf{t}$ and p is the point in $T_1 M$ that corresponds to the point where Γ enters T_w, let $h(p) = \mathbf{t}$. Note that the definition of $h(p)$ is unaffected by moving Γ by a deck transformation. To extend h to the required homeomorphism, send the open arc in $T_1 M$ that corresponds to $\Gamma \cap Int(S_w)$ to the open arc in the suspension of $(\hat{\Sigma}, \delta)$ that connects $\mathbf{t}$ to $\sigma(\mathbf{t})$. The resulting h is a homeomorphism that sends orbits of (X, g_t) to orbits of the suspension of $(\hat{\Sigma}, \delta)$, but does not preserve the time parameterization. However equation (5.1) shows that the time change cocycle (as in §1.5) is Lipschitz.

A more dynamical interpretation of the conjugacy can be given by using the arcs labeled a' and b' in Figure 5.2a to construct a cross section to the flow (X, g_t). The return map to the cross section will be topologically conjugate to the subshift $(\hat{\Sigma}, \delta)$. Thus (X, g_t) may be viewed as a variable time suspension of (or special flow over) $(\hat{\Sigma}, \delta)$.

§**5.3 Cocycles on the symbol spaces.** Since the dynamics of Φ restricted to Ω' are completely described by the symbolic system (Σ', δ) it is possible to translate the various cocycles of Section 2 into cocycles defined on (Σ', δ). The symbolic analog of the projection σ onto a shadowing geodesic (when it exists) is a map with image $\hat{\Sigma}$, the symbolic analog of the geodesic flow. In each case the symbolic analog of a function is indicated by the subscript s. The material of the last subsection implies that the symbolic analogs share all the relevant properties of their continuous counterparts.

The symbolic analog of the distance cocycle is the simplest to define. Let $D_s : \Sigma \times \mathbb{Z} \to \mathbb{Z}$ be given by $D_s(\mathbf{s}, n) = \ell(w(\mathbf{s}, n))$ (cf. [De]). The geometric interpretation of $D_s(\mathbf{s}, n)$ is the distance in $\mathcal{T}$ from the root to the position after n steps of a walk described by $\mathbf{s}$. It is clear that D_s is a subadditive cocycle for the shift δ on Σ' and that $D_s^*(\mathbf{s}) := \lim_{n\to\infty} D_s(\mathbf{s}, n)/n$ can be viewed as the asymptotic amount of cancellation of the infinite word $s_0 s_1 s_2 \ldots$.

To define σ_s, fix a sequence $\mathbf{s} \in \Sigma'$ and assume that $\tilde{x} \in \Omega_e$ with sequence $\mathbf{s}$ has a shadowing geodesic, or equivalently, that the walk with sequence $\mathbf{s}$ and root e has α and ω limits in S_∞, and these points are distinct. The image of the "lift" of σ_s should be the geodesic walk rooted at v_w which has these same limits on S_∞, where v_w is the vertex on the shadowing geodesic that is closest to v_e. If this walk has sequence $\mathbf{t} \in \hat{\Sigma}$, then $\sigma_s(\mathbf{s}) = \mathbf{t}$. The distance of $\tilde{x}$ from its shadowing geodesic is measured by the length of w, so define $\Delta_s(\mathbf{s}) = d_{\mathcal{T}}(v_e, v_w) = \ell(w)$. The cocycle that measures the distance of the orbit of a point from its shadowing geodesic is thus $B_s(\mathbf{s}, n) = \Delta_s(\delta^n(\mathbf{s})) - \Delta_s(\mathbf{s})$.

Algebraically, the map σ_s can be defined by cancellation. Think of $\mathbf{s} \in \Sigma'$ as a bi-infinite word written $l.r$ with l and r left and right infinite words, respectively. Reduce l starting from the decimal point and going left, and r by going right. The resulting bi-infinite word $l'.r'$ may have cancellations across the decimal point, but it may be written as $l''\bar{w}.wr''$ with w and $l''.r''$ reduced. If $l''.r''$ is not finite in either direction, then it represents a point on the shadowing geodesic and in fact $l''.r'' = \mathbf{t} = \sigma_s(\mathbf{s})$ with $\mathbf{t}$ as in the previous paragraph. The word w is also as in the previous paragraph, and so $\Delta_s(\mathbf{s}) = \ell(w)$.

§**5.4 Computing the rotation measure and properties of σ.** The diffeomorphism Φ restricted to the set Ω' is topologically conjugate to subshift Σ'. This subshift has uncountably many different ergodic invariant measures, but perhaps the most important is the *Parry measure* which maximizes the metric entropy, and in so doing makes it equal to the topological entropy. Since Σ' is a full shift on 4 symbols, the Parry measure, denoted

η', is the product measure. More specifically, if b is a finite allowable block of symbols and $C_{b,j}$ is the cylinder set of the block beginning at the j^{th} place, then $\eta'(C_{b,j}) = 1/4^n$, where n is the number of symbols in the block. The measure η' restricted to one-sided sequences in Σ' is also the stationary measure for the standard random walk on the Cayley graph of $F(a,b)$. In the language of §5.2, this random walk consists of all one-sided walks rooted at v_e using one-sided sequences from Σ'. Furstenberg showed that the typical such random walk has progressed a distance of $(1/2)n + o(n)$ after n steps (Section 4.2 of [Fu1], cf. exercise 9.1 in [PL]). Thus the asymptotic value of the distance cocycle for Parry measure is $D^*(\eta') = 1/2$.

The existence almost surely of limit points on S_∞ then follows from Lemma 2.1. The equivalent statement for the random walks is contained in Theorem 1.3 of [Fu2]. From this (or Lemma 2.2) the existence of the shadowing geodesic almost surely follows. Viewed algebraically this says that the typical bi-infinite word in the generators $a, b, \bar{a}, \bar{b}$ reduces to a bi-infinite word. It also says that the symbolic version of σ_s is defined almost everywhere with respect to η'.

The computation of the rotation measure of η' will follow from the computation of $(\sigma_s)_*(\eta')$. Given a pair of finite allowable blocks b_1, b_2 for $\hat{\Sigma}$, fix a pair of embedded arcs I_1 and I_2 in $\mathcal{T}$ that represent segments of walks with these blocks. If b_1 and b_2 have the same number of symbols, then there is an isometry of $\mathcal{T}$ that takes I_1 to I_2. In general, this isometry will include covering transformations as well as maps such as the interchange of a pair of subtrees rooted at the same vertex. Because the measure η' is symmetric in the symbols, the isometry induces a map from $\sigma_s^{-1}(C_{b_1,i})$ to $\sigma_s^{-1}(C_{b_2,j})$ for any i and j that preserves the η' measure. Thus all cylinder sets in $\hat{\Sigma}$ coming from blocks of the same length have the same $(\sigma_s)_*(\eta')$ measure. There are $4 \cdot 3^n$ different allowable blocks of length n in $\hat{\Sigma}$, thus $(\sigma_s)_*(\eta')(C_{b,i}) = 1/(4 \cdot 3^n)$. This is the same as the mass of the cylinder sets for the Parry measure $\hat{\eta}$ on $\hat{\Sigma}$, and so $(\sigma_s)_*(\eta') = \hat{\eta}$.

To compute the rotation measure of η', the measure $\hat{\eta}$ is connected with an invariant measure of the geodesic flow via the conjugacy h described in §5.2. Let μ be the unique, ergodic, g_t-invariant measure on X that is equivalent to $h_*(\hat{\eta})$. The geodesic flow restricted to X with the measure μ can be thought of as the geodesic flow on T with Liouville measure. Using $D^*(\eta') = 1/2$ and Theorem 4.1, $\rho(\eta') = (1/2)\mu$. Viewing $(\hat{X}, g_t)$ as the variable time suspension of $(\hat{\Sigma}, \delta)$ as in §5.2, the measure μ is the suspension of the Parry measure $\hat{\eta}$ on $\hat{\Sigma}$. The projection σ_s induces an almost everywhere defined map of the suspension of (Σ', δ) onto that of $(\hat{\Sigma}, \delta)$. This sends Parry measure to Parry measure, but the typical image orbit is moving half as fast as its preimage.

To compute the homology rotation of vector of η' (see §4.3), use the Abelianization of a and b as a basis for $H_1(T, \mathbb{Z})$. Since all the cylinder sets of length one in Σ' have equal mass, $S(x,1)$ takes the four values $(1,0), (-1,0), (0,1), (0,-1)$ on sets of equal measure. Thus $\int S(x,1)\, d\eta' = 0$, and so using Theorem 1.2, the homology rotation vector $S^*(\eta') = 0$. This corresponds to the well known fact that the standard walk on the Cayley graph of $\mathbb{Z}^2$ has mean progress zero. It also describes the statistics of the dynamics of Φ lifted to the $\mathbb{Z}^2$-cover T created by removing an equivariant family of open disks from the universal cover of the torus. In this cover, the orbits of Φ make no mean progress almost surely with respect to η'.

The behavior of Φ lifted to the universal cover manifests the frequently occurring dichotomy between what is dynamically typical in terms of topology and measure. By concatenating symbols in Σ' it is not difficult to construct a sequence such that the corresponding orbit under $\tilde{\Phi}$ beginning in Ω_e is dense in $\tilde{\Omega}'$, the full lift of Ω'. Since $\tilde{\Omega}'$ is a Baire space, a standard argument shows that a dense G_δ-set of points from $\tilde{\Omega}'$ also have this property. Thus in Ω' a dense-G_δ set of points have lifts whose orbits pass through every fundamental domain in the cover, and thus certainly do not have shadowing geodesics. This is in contrast to the full η'-measure set of points which do. Consequentially, the projection map σ is defined almost everywhere, but is not defined on a dense G_δ-set in Ω'.

It also is the case that σ is discontinuous at every point where it is defined in Ω'. To see this using σ_s, choose $\mathbf{s} \in \Sigma'$ so that $\sigma_s(\mathbf{s}) = \mathbf{t} \in \hat{\Sigma}$ exists. Let l and r be left infinite and right infinite reduced words, respectively, which begin with symbols different than t_{-1} and t_0, and are such that the concatenated sequence lr is also reduced. For $n \in \mathbb{N}$, let $b(n) = s_{-n}, s_{-n+1}, \ldots, s_{-1}, s_0, s_1, \ldots, s_n$ and $\mathbf{s}^{(n)} = lb(n).\bar{b}(n)r$, where as usual an overbar means the inverse but now applied to sequence blocks in the obvious fashion. Then $\mathbf{s}^{(n)} \to \mathbf{s}$, but $\sigma_s(\mathbf{s}^{(n)}) = l.r$ which is the same positive distance from $\sigma_s(\mathbf{s})$ for all n.

One may also use the symbolic models to show that for an η'-typical orbit, the distance from the shadowing geodesic is not bounded. This distance is $o(n)$ because Lemma 2.3 yields $B^*(\mathbf{s}) = 0$ almost surely. For each $m \in \mathbb{N}$, let $b(m) = a^m \bar{a}^m$, and so $\eta'(C_{b(m),0}) = 1/4^{2m}$. Since η' is ergodic under the shift, the generic point lands in $C_{b(m),0}$ about every 4^{2m} iterates. This ensures that for a generic point for the measure, there is always backtracking of all lengths that constantly happens along the walk corresponding to the orbit. Except in the special case that $\sigma_s(\mathbf{s})$ contains long blocks of $a's$ or $\bar{a}'s$, this means that the walk is wandering arbitrarily far away from its shadowing geodesic. More precisely, for any $m \in \mathbb{N}$, $B_s(\mathbf{s}, n) = m$ for infinitely many $n \in \mathbb{N}$. By changing the definition of $b(m)$ as needed, this can be made to happen for generic $\mathbf{s}$. An interesting question is a Central Limit Theorem for B_s: does $B_s(\mathbf{s}, n)/\sqrt{n}$ converges in law to a normal distribution with mean zero and positive standard deviation?

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