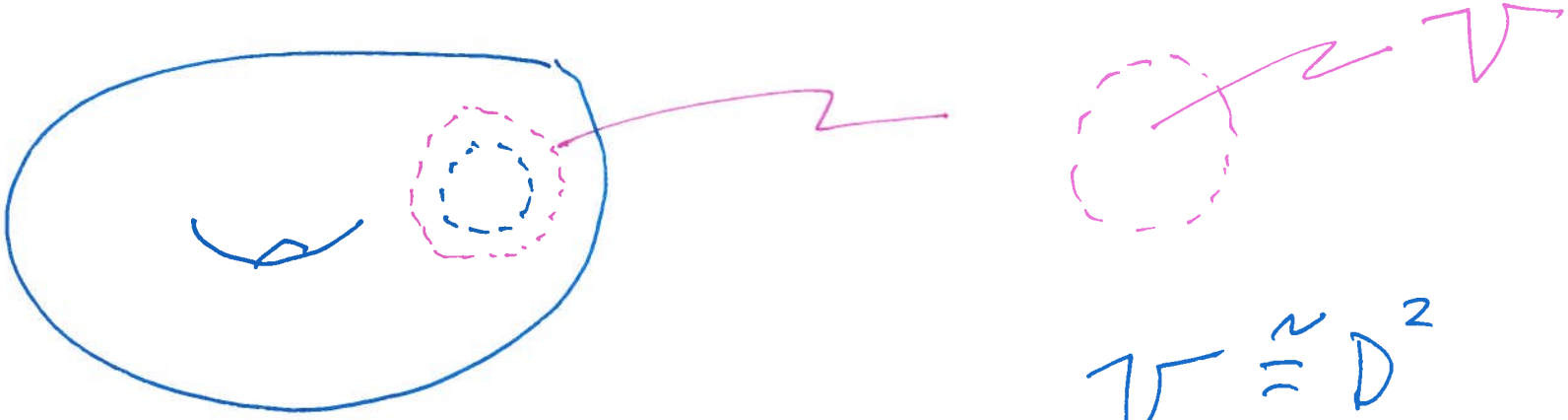
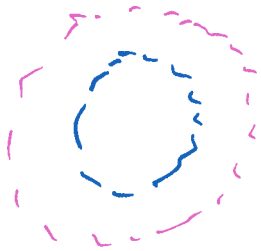
$\pi_1(T^2) \cong \mathbb{Z}^2$ via <u>SVC</u>



$V \cong D^2$

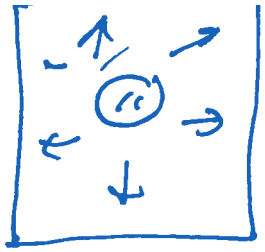
$U = T^2 -$ closed disk.

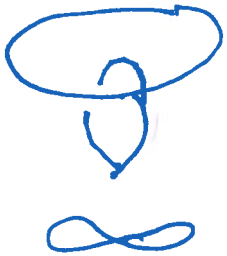


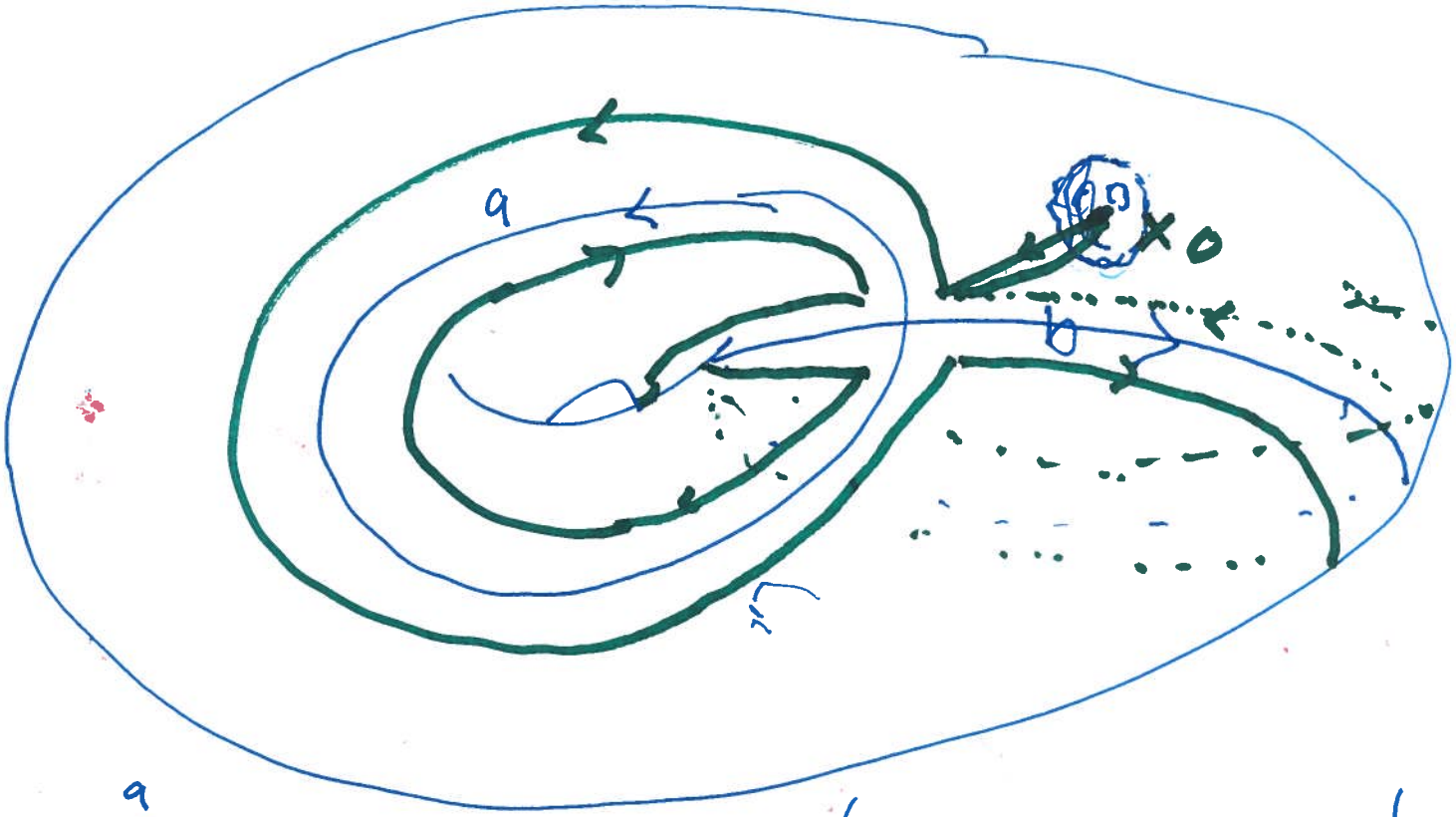
$U \cap V =$ annulus.

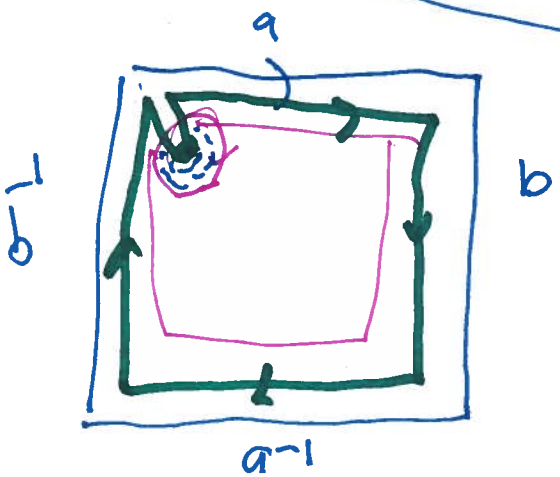
(N)

$$\pi_1(M) = \langle a, b \rangle$$



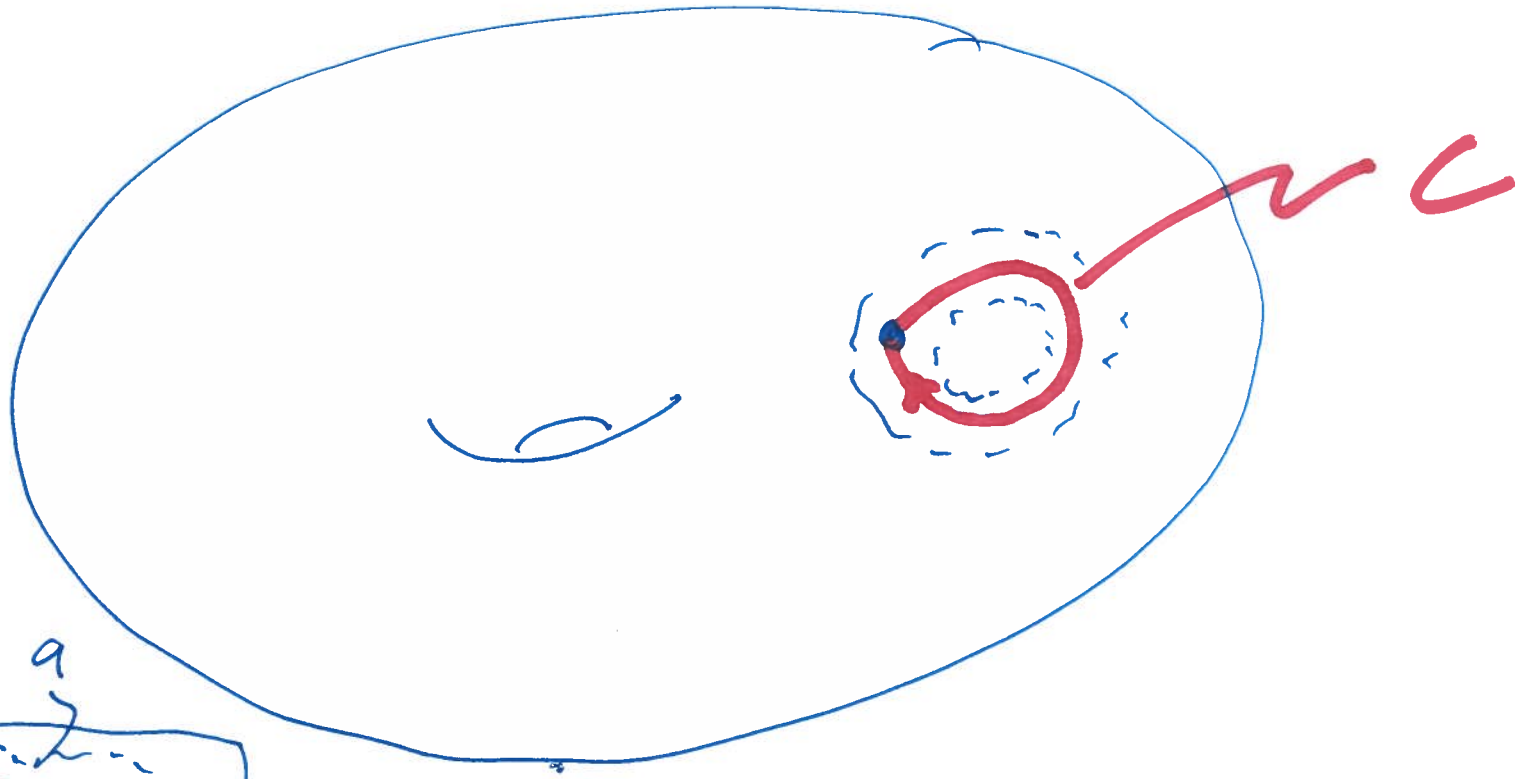


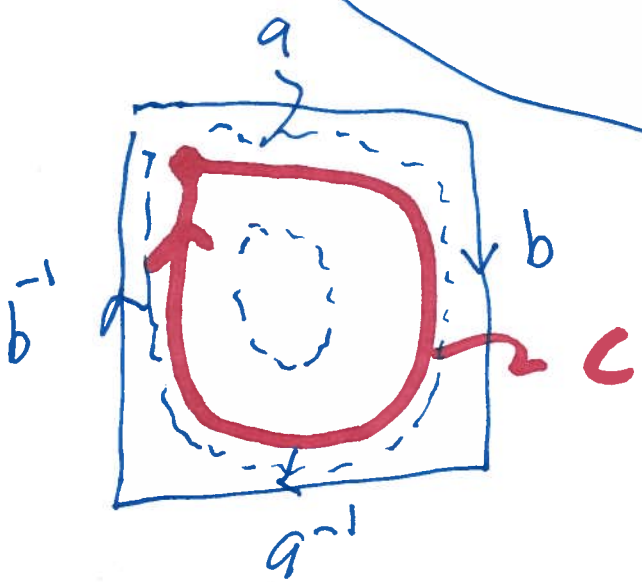




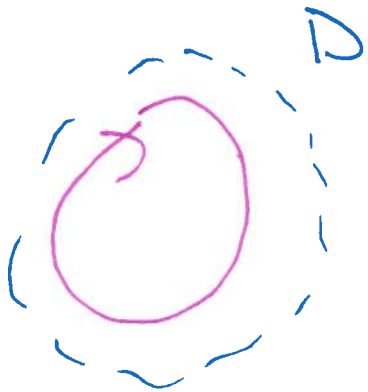
$$(L_4)_*(c) = aba^{-1}b^{-1}$$

(a)





$$(L_u)_* c = a b a^{-1} b^{-1}$$



$\pi_1(D) = e$

so $(i_V)_*(c) = e$

$\pi_1(U, x_0) = \langle a, b \rangle$

$\pi_1(V, x_0) = \langle e \rangle$ $\quad V$ contractible

$\pi_1(U \cap V, x_0) = \langle c \rangle$ $\quad U \cap V$ def. retracts to S^1

$(i_U)_*(c) = aba^{-1}b^{-1}$

$(i_V)_*(c) = e$

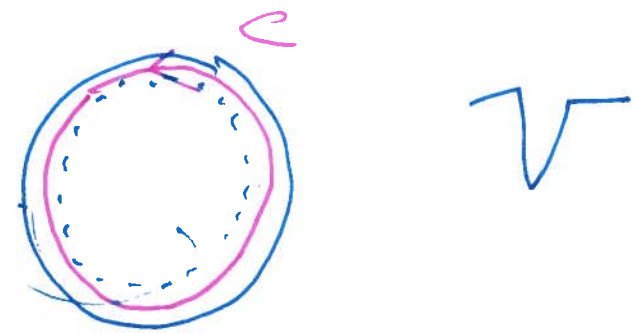
$\Rightarrow \pi_1(\pi^2) = \langle a, b \mid aba^{-1}b^{-1} \rangle = \mathbb{Z}^2$

$aba^{-1}b^{-1} \Leftrightarrow ab = ba$

Let X be the Möbius with a disk glued onto its single boundary component

U = Möbius band + a small ε nbd. of it

V = Disk + a small ε nbd of it.



$U \cap V \cong$ annulus

$U \simeq S^1$ def retract

$V \simeq pt$ def "

$U \cap V \simeq S^1$ def retract

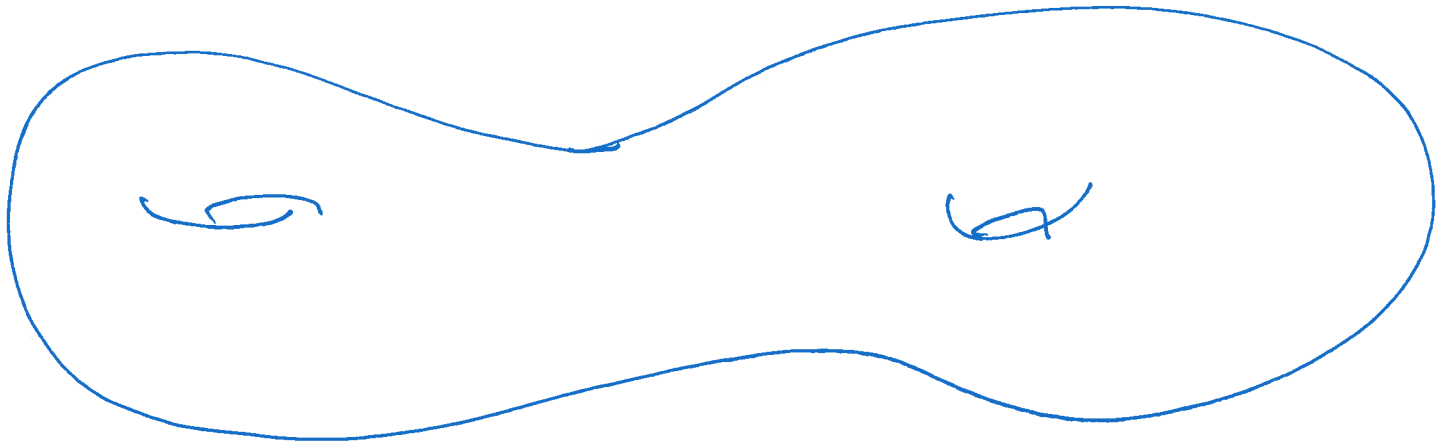
$\pi_1(U, x_0) = \langle a \rangle$

$\pi_1(V, x_0) = \langle e \rangle$

$\pi_1(U \cap V, x_0) = \langle c \rangle$

$(L_U)_*(c) = a^2 \quad (L_V)_*(c) = e$

$\pi_1(X, x_0) = \langle a \mid a^2 \rangle = \mathbb{Z}_2$

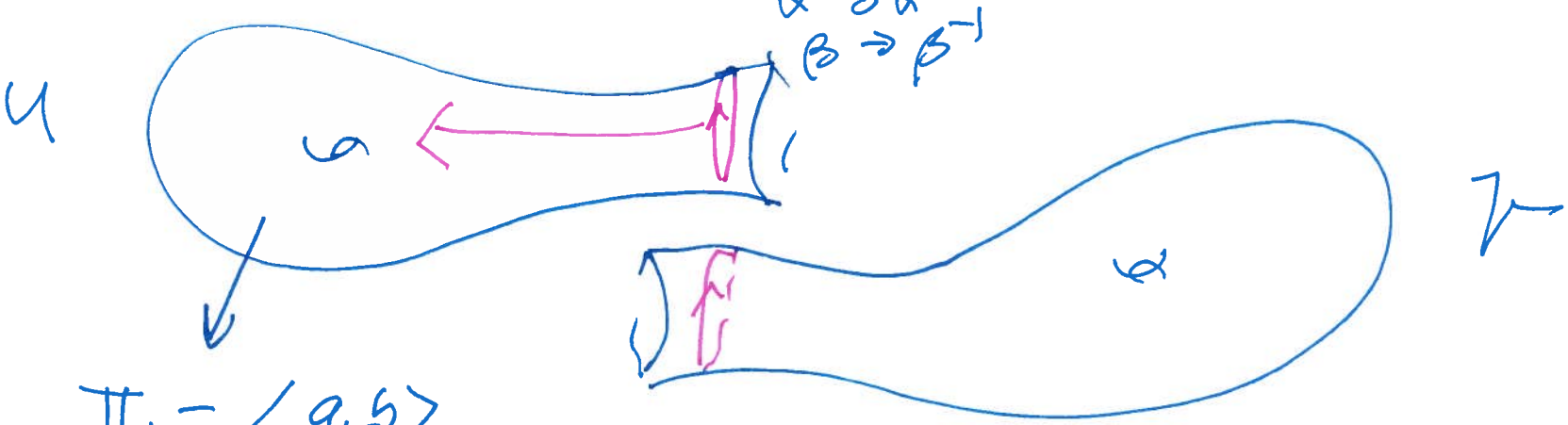


$$\pi_1(X, x_0) \overset{?}{=} \langle a, b, \alpha, \beta \mid aba^{-1}b^{-1}(\alpha\beta\alpha^{-1}\beta^{-1}) \rangle$$

$$= \langle a, b, \alpha, \beta \mid aba^{-1}b^{-1}\alpha\beta\alpha^{-1}\beta^{-1} \rangle$$

$\alpha \to \alpha^{-1}$

$\beta \to \beta^{-1}$



$\pi_1 = \langle a, b \rangle$

$\pi_1 \partial = \langle \alpha, \beta \rangle$

More Algebra

1st Abelian groups

Let $\{G_\alpha\}_{\alpha \in J}$ each $G_\alpha \subseteq G$ a subgroups

DEF The family generates G if each $g \in G$ can be expressed as a finite sum of elements $x_\alpha \in G_\alpha$ and $x = \underbrace{x_{\alpha_1} + \dots + x_{\alpha_n}}$

all α_i are distinct by combining terms.

write $G = \sum_{\alpha \in J} G_\alpha$

If this sum is unique write

$G = \bigoplus G_\alpha$ direct sum.

$$\mathbb{R}^{\mathbb{Z}} = \{(\ldots, x_{-2}, x_{-1}, x_0, x_1, \ldots) : x_i \in \mathbb{R}\}$$

is a group via

$$(\vec{x} + \vec{y})_i = x_i + y_i.$$

Let $G_n = \{(0, \ldots, x_n, 0, \ldots 0) : x_n \in \mathbb{R}\}$.

$$\mathbb{R}^{\infty} = \bigoplus_{n \in \mathbb{Z}} G_n \subsetneq \mathbb{R}^{\mathbb{Z}}$$

$$\bigoplus_{n \in \mathbb{Z}} \mathbb{R}$$

Lemma Given group G.

$\{G_\alpha\}_{\alpha \in J}$ subgroups and

$G = \bigoplus G_\alpha \Rightarrow$ If there is a family of homomorphs $h_\alpha : G_\alpha \to H$ for some Abelian group $H \Rightarrow$

$\exists!\, h : G \to H$ with $h \circ i_\alpha = h_\alpha$

or $h|_{G_\alpha} = h_\alpha$

$$\begin{tikzcd} G_\alpha \arrow[r, "h_\alpha"] \arrow[d, hook] & H \\ G \arrow[ur, dashed, "h"'] & \end{tikzcd} \qquad (*)$$

Conversely $G = \sum G_\alpha$ and (*) holds

$\Rightarrow G = \bigoplus G_\alpha$.

<u>Proof</u>: First converse, assume (*) holds.

Since $G = \sum G_\alpha$ given x, BYOC assume

$$x = \sum x_\alpha = \sum y_\alpha \quad \text{(finite sums)}$$

Pick an index β and $H = G_\beta$

and define family $\{h_\alpha\}$ via

$h_\alpha : G_\alpha \to H$ TRIV when $\alpha \neq \beta$

$h_\beta : G_\beta \to H = G_\beta$ is id

Since (*) let $h: G \to H$ be the extension guaranteed.

$$h(x) = h(\sum x_\alpha) = \sum h(x_\alpha) = x_\beta$$

$$h(x) = h(\sum y_\alpha) = \sum h(y_\alpha) = y_\beta$$

$\Longrightarrow$ $x_\beta = y_\beta$ and β was arbitrary.

Now for implication

$G = \bigoplus G_\alpha$. Given $\{h_\alpha\}$

if $x = \sum x_\alpha$ (finite sum!)

let $h(x) = \sum h_\alpha(x_\alpha)$.

(1) Since sum is finite, this is defined

(2) Since expression for x is unique, it is well defined

(3) It is obviously a homomorphism and makes the diagram commute.