Lect 2

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right] & B \arrow[r, "g"] & C \end{tikzcd}$$

$$g \circ f(x) = g(f(x))$$

$$id_A : A \to A \qquad id_A(a) = a$$

$f: A \to B$

(1) **injective**

or 1-1

- disjoint pts have disjoint images
- $\forall b \in (B)\ \exists!\ a$ with $b = f(a)$

image exists unique

use in proofs →
- $\underline{\text{if } f(a) = f(a') \Rightarrow a = a'}$

- Surjective or onto

  $\text{range}(f) = B$

  $\forall b \in B \ \exists a$ with $f(a) = b$

- bijective if both surjective and injective.

  In this case $\exists f^{-1} : B \to A$ with $f^{-1} \circ f = \mathrm{id}_A$ $\ f \circ f^{-1} = \mathrm{id}_B$

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

Lemma $f: A \to B$

$g: B \to A$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd}$$

$f \circ g = id_B$

$\Rightarrow$ f is surjective

g is injective.

Proof g injective: Assume $g(b) = g(b')$

$\Rightarrow f \circ g(b) = f \circ g(b')$

$\Rightarrow id_B(b) = id_B(b')$

$\Rightarrow b = b'$ done

f is Surjective!

$$f \circ g(b) = id_B(b) = b$$

$$f(g(b)) = b$$

So b is the image of $g(b)$.

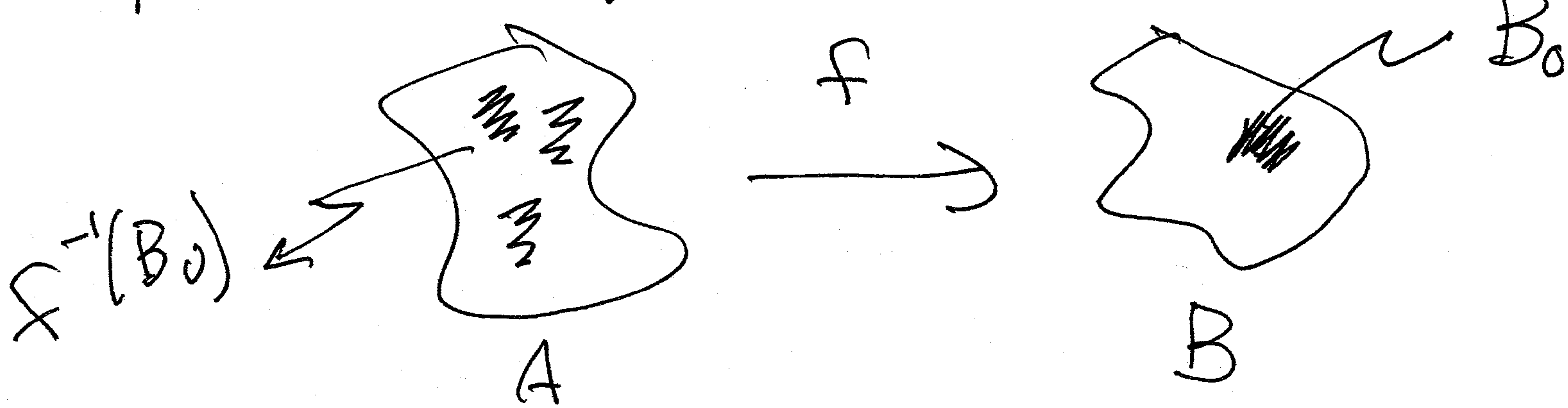
__Cor__ If $f: A \to B$ $g: B \to A$

__and__ $f \circ g = \mathrm{id}_B$ $g \circ f = \mathrm{id}_A$

$\Rightarrow$ f and g are each bijective

and $g = f^{-1}$ and $f = g^{-1}$

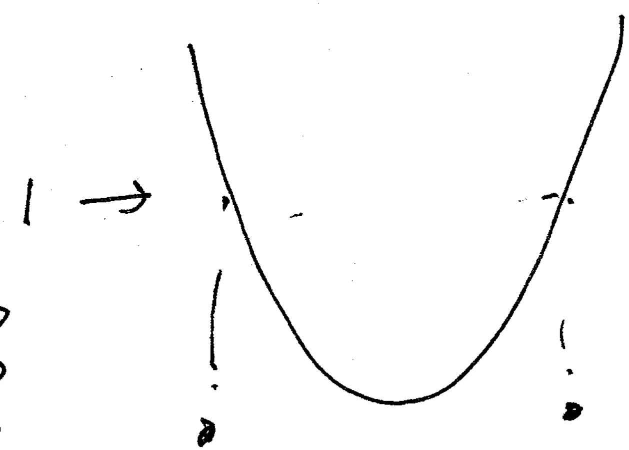
Pre Image of a set ~~of~~



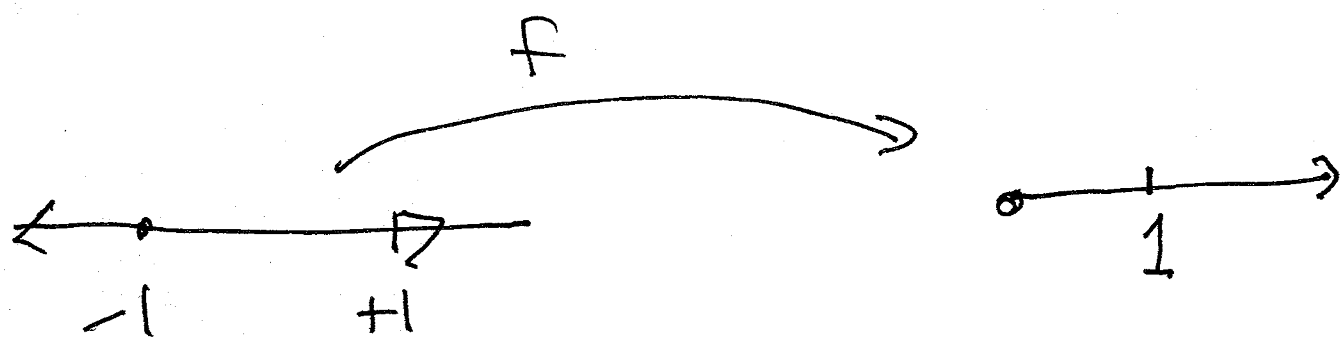
Def $f: A \to B \quad B_0 \subseteq B$

$$f^{-1}(B_0) = \{a \in A : f(a) \in B_0\}$$

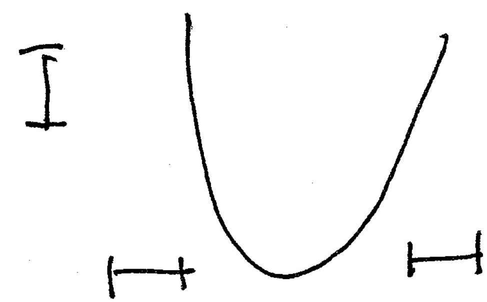
$f(x) = x^2 \quad f: \mathbb{R} \to \mathbb{R}$



$$f^{-1}(\{1\}) = \{-1, 1\}$$



$$f^{-1}([1,4]) = [-2,-1] \cup [1,2]$$



Interactions of functions and set operations.

$\boxed{\cup, \cap, -, \times, \subseteq}$

$$A_0 \subseteq A \qquad B_0 \subseteq B$$

$$f(A_0) \subseteq f(A) \qquad f^{-1}(B_0) \subseteq f^{-1}(B)$$

Inverse functions work more nicely with set algebra than functions

---

eg// (a) $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$

$\cancel{f(B_0 \cap B_1) \neq}$

(b) $f(A_0 \cap A_1) \neq f(A_0) \cap f(A_1)$ in general

(c) $f(A_0 \cap A_1) \subseteq f(A_0) \cap f(A_1)$ in general.

Proof of (a) $f^{-1}(B_0 \wedge B_1) = f^{-1}(B_0) \wedge f^{-1}(B_1)$

$a \in LHS.$ so $f(a) \in B_0 \wedge B_1$

<u>so</u> $f(a) \in B_0$ and $f(a) \in B_1$

so $a \in f^{-1}(B_0)$ $a \in f^{-1}(B_1)$

so $a \in f^{-1}(B_0) \wedge f^{-1}(B_1) = RHS.$

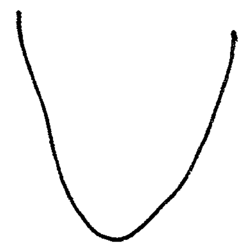
$a \in RHS.$ $\begin{matrix} a \in f^{-1}(B_0) \\ a \in f^{-1}(B_1) \end{matrix} \Rightarrow \begin{matrix} f(a) \in B_0 \\ f(a) \in B_1 \end{matrix}$

so $f(a) \in B_0 \wedge B_1$

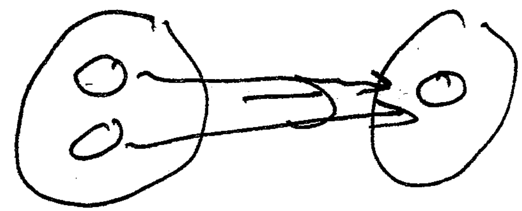
so $a \in f^{-1}(B_0 \wedge B_1) = LHS.$

(b) Counterexample

$$f(A_0 \wedge A_1) \neq f(A_0) \wedge f(A_1)$$

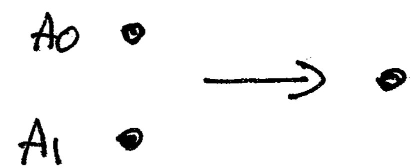


$$f(x) = x^2$$



$$A_0 = [-2, -1]$$

$$A_1 = [1, 2]$$



$$A_0 \wedge A_1 = \emptyset$$

Skip proof of (c)

$$f(A_0) = [1, 4] = f(A_1)$$

$$\text{so } f(A_0) \wedge f(A_1) = [1, 4] \neq \emptyset = f(\emptyset) = f(A_0 \wedge A_1)$$

Relations in $\mathbb{Z}$

$n \sim m$ if $n-m$ is divisible by 3

$\sim -3 \sim 0 \sim 3 \sim 6 \sim$

$\sim -2 \sim 1 \sim 4 \sim 7 \sim$

$\sim -1 \sim 2 \sim 5 \sim 8 \sim$

$(0,3) \in C$
$(3,6) \in C$
$(1,4) \in C$
$\vdots$

A relation on A is a subset $C \subseteq A \times A$ if $(a_1, a_2) \in C$

write $a_1 C a_2$

Equivalence Relation satisfies

(1) reflexive $x C x \quad \forall x \in A$

(2) Symmetric $x C y \Rightarrow y C x$

(3) Transitive $x C y$ and $y C z$
$\Rightarrow x C z$.

Usually written $x \sim y$
when $(x, y) \in C$.

---

equivalence classes

$$[x] = \{ y \in A : x \sim y \}$$

for $n-m$ div by 3 there are

3 equiv classes.

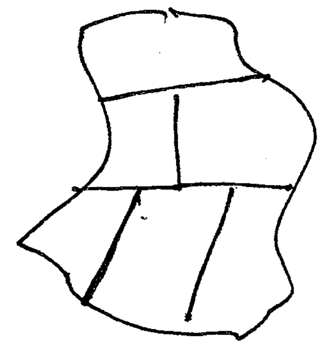
$$\boxed{\begin{array}{l} x \sim y \\ \iff x \equiv y \bmod 3 \end{array}}$$

$[0], [1], [2]$

partition $\mathbb{Z}$ into 3

disjoint subsets.

Theorem equivalences classes are either disjoint or equal so they partition the space into disjoint subsets



or If ~~[x]~~

$$[x] \wedge [y] \neq \phi \Rightarrow [x] = [y]$$

Proof: say $z \in [x] \wedge [y] \Rightarrow y \sim z$

$x \sim z \Rightarrow z \sim x$

transitivity $y \sim x$

Next we show $y \sim x \Rightarrow \underline{\underline{[y]=[x]}}$

Say $z \in [x] \Rightarrow x \sim z$ and $x \sim y$

$\Rightarrow y \sim z \Rightarrow z \in [y]$

So $[x] \subseteq [y]$

Switch the roles of y and x

yields $[y] \subseteq [x]$

So $[x] = [y]$. $\blacksquare$

DEF: Partition of X is a disjoint collection of subsets $X_\lambda \subseteq X$ with $\lambda \in \Lambda$ with $X = \bigcup X_\lambda$

FACT: If $\sim$ is an equiv. rel on A then the equiv classes yield a partition of A

FACT: If $\{A_\lambda\}$ is a partition of A define $a \sim a' \iff a$ and a' are in the same $A_\lambda \Rightarrow \sim$ is an equiv rel.