~~equivalence~~ relation on A is

a subset $C \subseteq A \times A$.

Two main type of relations

1. equivalence

$$x C x$$

$$x C y \Rightarrow y C x$$

$$x C y \quad y C z \Rightarrow x C z$$

Partition the space into equiv classes.

2 order relations

2 main types

A Strict simple (linear, total) order. or just "order"

Strict → (1) $x \neq y \Rightarrow$ either $x < y$ or $y < x$

(2) $x < x$ never happens

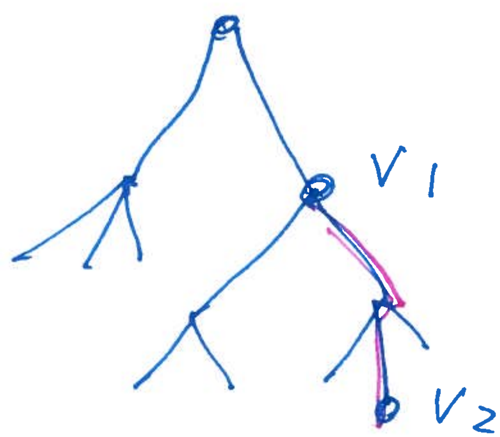
(3) $x < y, y < z \Rightarrow x < z$

eg. $(\mathbb{R}, <)$ $\boxed{\text{written } x < y}$

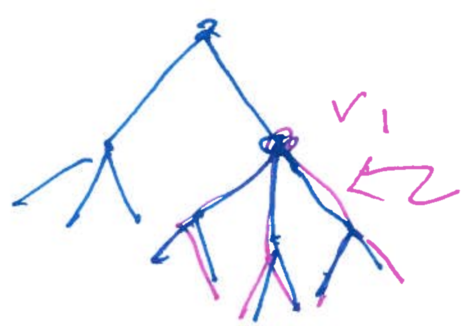
B) Strict partial order

(2) xCx never happens

(3) Transitivity.



vertex v_1 > vertex v_2 if it is above it and connected by a downward path.



v_1 related to tree below.

Given a strict order (A) or (B)

define $a \leq b$ if $a < b$ or $a = b$

yields

(A') linear order

(1) $\forall x, y$ either $x \leq y$ or $y \leq x$

(2) $x \leq x$

(3) $x \leq y$ and $y \leq x \Rightarrow x = y$

(4) transitivity

(B') partial order:

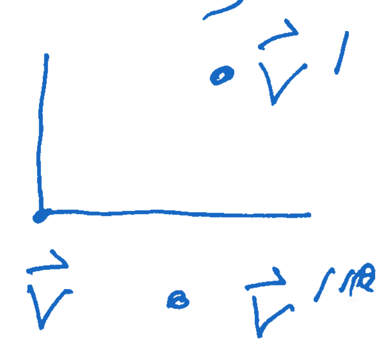
just (2), (3), (4)

---

example of a partial order on $\mathbb{R}^2$ $\quad \vec{v} = (a,b) \quad \vec{v}\,' = (a',b')$

$\vec{v} \geq \vec{v}\,'$ if $a - a' \geq 0$ and $b - b' \geq 0$

<u>or</u> equivalently $\vec{v} - \vec{v}\,' \in$ closed first quadrant.

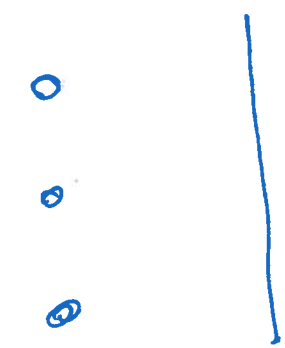


$\vec{v}\,''$ not related to $\vec{v}$

Is there a linear order on $\mathbb{R}^2$? (4)

Focus on strict linear orders

$\mathbb{Z}$ $\mathbb{R}$



open order interval

$a \neq b$

$(a,b) = \{x : a < x < b\}$.

$(a,\infty) = \{x : a < x\}$

if (a,b) is empty then

b is the immediate successor to a

and a " " " predecessor to b.

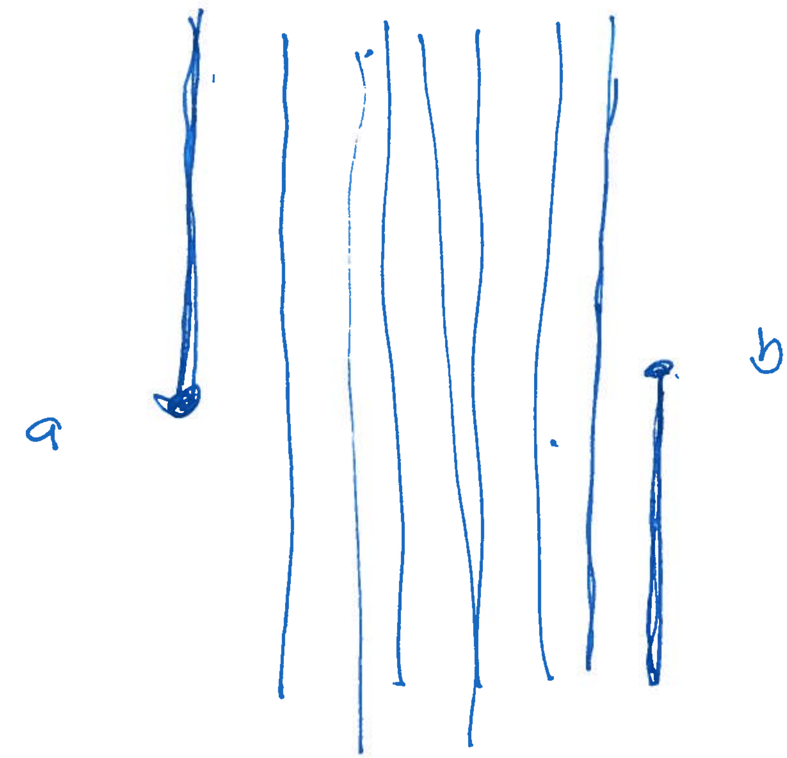
~~Dist~~ Dictionary order on $A \times B$

(~~spelling~~ alphabetical order
or lexicographic order)

<u>DEF</u> Given $(A, <_A)$, $(B, <_B)$
strict linear orders. Define $<$ on $A \times B$
as $(a_1, b_1) < (a_2, b_2)$
<u>if</u> $a_1 < a_2$ <u>OR</u> $a_1 = a_2$ and $b_1 < b_2$

<u>HW</u>: $(A \times B, <)$ is a strict linear order

On $\mathbb{R} \times \mathbb{R}$ (a, b)



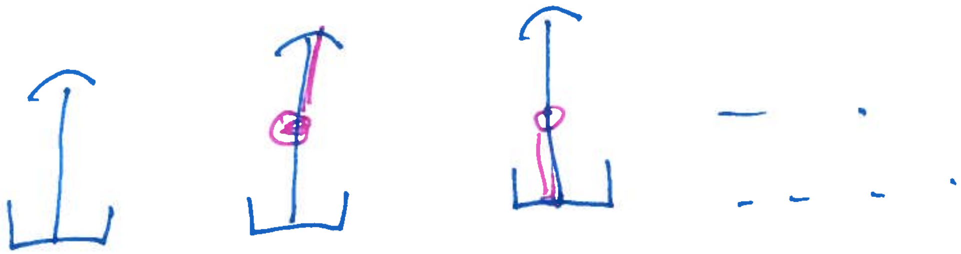
DEF:

DEF: $(A, <_A)$ and $(B, <_B)$ have the same ~~ordering~~ structure if $\exists$ bijection $\phi: A \to B$ so that $a_1 <_A a_2 \Rightarrow \phi(a_1) <_B \phi(a_2)$.
also said to be <u>order isomorphic</u>

RK: Def implies $b_1 <_B b_2 \Rightarrow \phi^{-1}(b_1) <_A \phi^{-1}(b_2)$

Proof Take the contrapositive
not $(\phi(a_1) <_B \phi(a_2))$ is (either $\phi(a_1) = \phi(a_2)$
$\phi(a_2) < \phi(a_1))$

$\mathbb{Z}_+ \times [0,1)$ with dictionary order



eg// $\mathbb{Z}_+ \times [0,1)$ is order isomorphic to $([0,\infty), <)$

$$\phi(n,t) = n + t - 1$$

$\sup([0,1)) = 1$

but doesn't have a largest element since $1 \in [0,1)$

$(A, <)$ has the l.u.b. property if every $A_0 \subseteq A$ that has an upper bound has a l.u.b.

$(\mathbb{R}, <)$ has the l.u.b. property