This HW uses material from Lectures 1-6. Remember that I will be grading these on-screen, so make sure your pdf is clear and legible. Also, please do the problems in order with the problem number labeled clearly.

1. Prove that for any sets $A$ and $B$,
   \[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C). \]

2. Determine if the statement is true or false. If the statement is true, give a proof. If the statement is false, give a counterexample.
   
   (a) For $f : A \to B$ and $A_0, A_1 \subset A$,
       \[ f(A_0 \cup A_1) = f(A_0) \cup f(A_1). \]
   (b) For any sets $A$ and $B$,
       \[ A - (B - A) = A - B. \]
   (c) For any sets $V_1, W_1, V_2, W_2$,
       \[ (V_1 \times W_1) \cup (V_2 \times W_2) = (V_1 \cup V_2) \times (W_1 \cup W_2). \]
   (d) If $A \times B$ is finite then both $A$ and $B$ are finite.
   (e) If $A$ and $B$ are nonempty and $A \times B$ is finite then both $A$ and $B$ are finite.

3. Suppose that $X$ is a set and \( \{ A_\alpha : \alpha \in I \} \) is an indexed family of sets (which contains at least one set). Prove that
   \[ X - \bigcup_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I} (X - A_\alpha). \]

4. Define a relation $S$ on the set of real numbers $\mathbb{R}$ by
   \[ S = \{(a, b) \in \mathbb{R} \times \mathbb{R} : a - b \text{ is an integer}\}. \]
   Prove that $S$ is an equivalence relation on $\mathbb{R}$.

5. Suppose that $<_A$ is a strict linear order on $A$, and $<_B$ is a strict linear order on $B$. Prove that dictionary order relation on $A \times B$ is a strict linear order on $A \times B$.

6. Suppose that $<_A$ is a strict linear order on $A$, and $<_B$ is a strict linear order on $B$ and there exists a bijection $\phi : A \to B$ so that
   \[ a <_A a' \text{ implies } \phi(a) <_B \phi(a'). \]
   Show that
   \[ \phi(a) <_B \phi(a') \text{ implies } a <_A a'. \]
7. Assuming that the set of real numbers \( \mathbb{R} \) has the least upper bound property show that it has the greatest lower bound property.

8. Suppose that \( A \) and \( B \) are sets. Suppose that \( A \) is countable, and there is a surjective function \( f : A \to B \). Prove that \( B \) is countable.

9. Prove that the set of rational numbers is countable.

10. A real number is said to be algebraic if and only if it satisfies some polynomial equation of positive degree of the form

\[
a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0,
\]

where each \( a_i \) is an integer. Assuming that a degree \( n \) polynomial has at most \( n \) distinct roots, prove that the set of all algebraic numbers is countable.