Topology HW 1 • FALL 2021 • PROF. BOYLAND

This HW uses material from Lectures 1-6. Remember that I will be grading these on-screen, so make sure your pdf is clear and legible. Also, please do the problems in order with the problem number labeled clearly.

1. Prove that for any sets A and B,
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

2. Determine if the statement is true or false. If the statement is true, give a proof. If the statement is false, give a counterexample.

   (a) For $f : A \to B$ and $A_0, A_1 \subset A$,
   $$f(A_0 \cup A_1) = f(A_0) \cup f(A_1).$$

   (b) For any sets A and B,
   $$A - (B - A) = A - B.$$

   (c) For any sets V_1, W_1, V_2, W_2,
   $$(V_1 \times W_1) \cup (V_2 \times W_2) = (V_1 \cup V_2) \times (W_1 \cup W_2).$$

   (d) If $A \times B$ is finite then both A and B are finite.

   (e) If A and B are nonempty and $A \times B$ is finite then both A and B are finite.

3. Suppose that X is a set and $\{A_\alpha : \alpha \in I\}$ is an indexed family of sets (which contains at least one set). Prove that
$$X - \bigcup_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I} (X - A_\alpha).$$

4. Define a relation S on the set of real numbers $\mathbb{R}$ by
$$S = \{(a, b) \in \mathbb{R} \times \mathbb{R} \,:\, a - b \text{ is an integer}\}.$$
   Prove that S is an equivalence relation on $\mathbb{R}$.

5. Suppose that $<_A$ is a strict linear order on A, and $<_B$ is a strict linear order on B. Prove that dictionary order relation on $A \times B$ is a strict linear order on $A \times B$.

6. Suppose that $<_A$ is a strict linear order on A, and $<_B$ is a strict linear order on B and there exists a bijection $\phi : A \to B$ so that
$$a <_A a' \text{ implies } \phi(a) <_B \phi(a').$$
   Show that
$$\phi(a) <_B \phi(a')) \text{ implies } a <_A a'.$$

7. Assuming that the set of real numbers $\mathbb{R}$ has the least upper bound property show that it has the greatest lower bound property.

8. Suppose that A and B are sets. Suppose that A is countable, and there is a surjective function $f : A \to B$. Prove that B is countable.

9. Prove that the set of rational numbers is countable.

10. A real number is said to be algebraic if and only if it satisfies some polynomial equation of positive degree of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

where each a_i is an integer. Assuming that a degree n polynomial has at most n distinct roots, prove that the set of all algebraic numbers is countable.