TOPOLOGY HW 2.4 • SPRING 2022 • PROF. BOYLAND

This HW uses material from Lectures 19-22. This HW is worth 15 points, half the usual.

1. Assume $X$ is path connected.
   (a) If $\phi : \pi_1(X, x_0) \to \pi_1(X, x_1)$ is an isomorphism show that it induces an isomorphism
   $$\phi' : \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)] \to \pi_1(X, x_1)/[\pi_1(X, x_1), \pi_1(X, x_1)].$$
   (b) Recall that for a path $\alpha$ from $x_0$ to $x_1$ the induced isomorphism on fundamental groups is denoted $\hat{\alpha}$. For any two paths $\alpha, \beta$ from $x_0$ to $x_1$ show that the induced maps on the Abelianizations are the same, i.e. $\hat{\alpha}' = \hat{\beta}'$.
2. Given an equivalence relation on $X$ let $Y = X/\sim$ with the quotient topology and quotient map $p : X \to Y$. Let $Z$ be a topological space and $g : X \to Z$ a map that is constant on each set $p^{-1}(y)$.
   (a) Show that there exists a map $f : Y \to Z$ so that $f \circ p = g$.
   (b) Show that $f$ is continuous if and only if $g$ is continuous.
3. If $M$ is a compact surface, show that $\pi_1(M \# S^2) \cong \pi_1(M)$
4. (a) For each $n > 1$, construct a space $X$ with $\pi_1(X) \cong \mathbb{Z}_n$.
   (b) Construct a space $X$ with $\pi_1(X) \cong \mathbb{Z}_3 * \mathbb{Z}_5$.
5. Consider the octagon with labeling $abcda^{-1}b^{-1}c^{-1}d^{-1}$.
   (a) Show that identifying the edges according to the labeling yields a compact surface.
   (b) What surface is it? (be sure to prove your result).