This HW uses material from Lectures 6-11.

1. Let \( W \) consist of all the functions from \( \mathbb{Z}_+ \to \{1, 2\} \). Show that 
\[
\text{card}(W) = \text{card}(\mathcal{P}(\mathbb{Z}_+)) = \text{card}(\{1, 2\}^{\mathbb{Z}_+}).
\]

2. Let \( X \) be a set, and let \( \mathcal{T} \) be the collection of all subsets \( A \) of \( X \) such that \( X - A \) is either countable or all of \( X \). Prove that \( (X, \mathcal{T}) \) is topological space.

3. Prove that the dictionary order topology on \( \mathbb{R} \times \mathbb{R} \) is the same as the product topology on \( \mathbb{R}_d \times \mathbb{R} \), where \( \mathbb{R}_d \) denotes the set of real numbers with the discrete topology.

4. Let \( K = \{1/n : n \in \mathbb{Z}_+\} \). Let \( \mathcal{B}_K \) be the set of all intervals \((a, b) \subset \mathbb{R}\) along with all sets of the form \((a, b) - K\).
   
   (a) Show that \( \mathcal{B}_K \) is a base. Let the topology on \( \mathbb{R} \) generated by \( \mathcal{B}_K \) be denoted \( \mathcal{T}_K \).
   
   (b) Prove that \( \mathcal{T}_K \) is strictly finer than the standard topology on \( \mathbb{R} \).
   
   (c) Prove that \( \mathcal{T}_K \) and the lower limit topology are not comparable.

5. Let \( \mathcal{B} \) denote the collection of all half-open intervals, \([a, b) \) where \( a \) and \( b \) are rational numbers with \( a < b \). Prove that \( \mathcal{B} \) is a basis for a topology on \( \mathbb{R} \), and the topology generated by \( \mathcal{B} \) is different from the lower limit topology.

6. Let \( X \) be a topological space. Let \( D = \{(x, y) \in X \times X : x = y\} \). Prove that \( X \) is a Hausdorff space if and only if \( D \) is a closed subset of \( X \times X \), where \( X \times X \) has the product topology.

7. Let \( X \) be a topological space. Suppose that \( A \) and \( B \) are open subsets of \( X \) with \( \overline{A} = X = \overline{B} \). Prove that \( \overline{A \cap B} = X \).

8. Define a collection \( \mathcal{T} \) of subsets of \( \mathbb{Z}_+ \) as follows:
   
   \( W \in \mathcal{T} \) if and only if \( n \in W \) implies that all positive divisors of \( n \) are also elements of \( W \).
   
   Verify that \( \mathcal{T} \) is a topology on \( \mathbb{Z}_+ \). In this topology find \( \{1\} \) and \( \{2\} \).

9. Consider the space \( \mathbb{Z}_+ \) with the finite complement topology. Is this space a Hausdorff space? Is this space a \( T_1 \) space?

10. Determine if the statement is true of false. If true, give a proof. If false give a counterexample with proof.
    
    If \( X \) is a topological space, and \( W \) is an open subset of \( X \), then \( W = \text{Int}(W) \).

11. Prove that every strict linearly ordered set with the order topology is a Hausdorff space.

12. Suppose that \( A_\alpha \) is an indexed family of subsets of a topological space \( X \). Prove that 
\[
\bigcup \overline{A_\alpha} \subset \overline{\bigcup A_\alpha}.
\]
   
   Give an example with \( X = \mathbb{R} \) to show that equality need not hold.