TOPOLOGY HW 2 • FALL 2021 • PROF. BOYLAND

This HW uses material from Lectures 6-11.

1. Let W consist of all the functions from $\mathbb{Z}_+ \to \{1,2\}$. Show that
$$\operatorname{card}(W) = \operatorname{card}(\mathcal{P}(\mathbb{Z}_+)) = \operatorname{card}(\{1,2\}^{\mathbb{Z}_+}).$$
2. Let X be a set, and let $\mathcal{T}$ be the collection of all subsets A of X such that $X - A$ is either countable or all of X. Prove that $(X, \mathcal{T})$ is topological space.
3. Prove that the dictionary order topology on $\mathbb{R} \times \mathbb{R}$ is the same as the product topology on $\mathbb{R}_d \times \mathbb{R}$, where $\mathbb{R}_d$ denotes the set of real numbers with the discrete topology.
4. Let $K = \{1/n \colon n \in \mathbb{Z}_+\}$. Let $\mathcal{B}_K$ be the set of all intervals $(a,b) \subset \mathbb{R}$ along with all sets of the form $(a,b) - K$.
   (a) Show that $\mathcal{B}_K$ is a base. Let the topology on $\mathbb{R}$ generated by $\mathcal{B}_K$ be denoted $\mathcal{T}_K$.
   (b) Prove that $\mathcal{T}_K$ is strictly finer than the standard topology on $\mathbb{R}$.
   (c) Prove that $\mathcal{T}_K$ and the lower limit topology are not comparable.
5. Let $\mathcal{B}$ denote the collection of all half-open intervals, $[a,b)$ where a and b are rational numbers with $a < b$. Prove that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$, and the topology generated by $\mathcal{B}$ is different from the lower limit topology.
6. Let X be a topological space. Let $D = \{(x,y) \in X \times X : x = y\}$. Prove that X is a Hausdorff space if and only if D is a closed subset of $X \times X$, where $X \times X$ has the product topology.
7. Let X be a topological space. Suppose that A and B are open subsets of X with $\overline{A} = X = \overline{B}$. Prove that $\overline{A \cap B} = X$.
8. Define a collection $\mathcal{T}$ of subsets of $\mathbb{Z}_+$ as follows:

   $W \in \mathcal{T}$ if and only if $n \in W$ implies that all positive divisors of n are also elements of W.

   Verify that $\mathcal{T}$ is a topology on $\mathbb{Z}_+$. In this topology find $\overline{\{1\}}$ and $\overline{\{2\}}$.
9. Consider the space $\mathbb{Z}_+$ with the finite complement topology. Is this space a Hausdorff space? Is this space a T_1 space?
10. Determine if the statement is true of false. If true, give a proof. If false give a counterexample with proof.

    If X is a topological space, and W is an open subset of X, then $W = Int(\overline{W})$.
11. Prove that every strict linearly ordered set with the order topology is a Hausdorff space.
12. Suppose that A_α is an indexed family of subsets of a topological space X. Prove that
$$\bigcup \overline{A_\alpha} \subset \overline{\bigcup A_\alpha}.$$
    Give an example with $X = \mathbb{R}$ to show that equality need not hold.