TOPOLOGY HW 3 • FALL 2021 • PROF. BOYLAND

This HW uses material from Lectures 11-18a.

1. Say $A \subset X$ is dense if $\overline{A} = X$.
   (a) Show that A is dense in X if and only if every nonempty open subset V in X satisfies $V \cap A \neq \emptyset$.
   (b) Assume that X and Y are topological spaces with Y Hausdorff and A is dense in X. Suppose that $f : X \to Y$ and $g : X \to Y$ are continuous functions with $f(a) = g(a)$ for all $a \in A$. Prove that $f(x) = g(x)$ for all $x \in X$.
2. A is a subset of the topological space X.
   (a) Show that $x \in \mathrm{Int}(A)$ if and only if there is an open set U with $x \in U \subset A$.
   (b) Let the boundary of A be $\mathrm{Bd}(A) = \overline{A} \cap \overline{(X - A)}$. Show that $x \in \mathrm{Bd}(A)$ if and only if every open set V with $x \in V$ contains points of both A and $X - A$.
   (c) Prove that $\mathrm{Bd}(A) \cap \mathrm{Int}(A) = \emptyset$ and that $\overline{A} = \mathrm{Int}(A) \cup \mathrm{Bd}(A)$.
3. Consider $\mathbb{Z}_+$ with the finite complement topology. Determine if the following sequences converge and if so, to which point or points.
   (a) $x_n = 2n + 3$
   (b) $x_n = 3 + (-1)^n$
4. Recall that two topological spaces X and Y are homeomorphic if and only if there is a homeomorphism $h : X \to Y$. Suppose that $\{X_\lambda : \lambda \in \Lambda\}$ and $\{Y_\lambda : \lambda \in \Lambda\}$ are indexed families of topological spaces with X_λ homeomorphic to Y_λ for each $\lambda \in \Lambda$. Prove that $\prod_{\lambda \in \Lambda} X_\lambda$ and $\prod_{\lambda \in \Lambda} Y_\lambda$ are homeomorphic. Use the product topology on the product spaces.
5. Assume that d and d' are metrics on X and that there are positive constants c_1, c_2 with
$$c_1 d(x, y) \leq d'(x, y) \leq c_2 d(x, y)$$
   for all $x, y \in X$. Show that d and d' induce the same topology.
6. We showed in class that on $\mathbb{R}^{\mathbb{Z}_+}$ the box topology is finer than the uniform topology which in turn is finer than the product topology. Give examples that show that the box topology is *strictly* finer than the uniform topology which in turn is *strictly* finer than the product topology. You can use the fact that the product topology is induced by the metric D.
7. Give $X^{\mathbb{Z}_+}$ the product topology and let $\{\underline{x}_n\}$ be a sequence in $X^{\mathbb{Z}_+}$.
   (a) Show that $\underline{x}_n \to \underline{x}$ if and only if for each $i \in \mathbb{Z}_+$, $\pi_i(x_n) \to \pi_i(x)$. In other words, a sequence converges if and only if all its components converge.
   (b) Is this result true when we give $X^{\mathbb{Z}_+}$ the box topology?

8. Let (X, d) be a metric space.

   (a) Show that $d : X \times X \to \mathbb{R}$ is continuous where $X \times X$ is given the product topology.

   (b) If the sequences $x_n \to x$ and $y_n \to y$ convergence in X show that the sequence of real numbers $d(x_n, y_n) \to d(x, y)$.

9. Given metric spaces (X_i, d_i) for $i = 1, \ldots, n$ show that

$$\rho(x, y) = \max\{d_1(x, y), \ldots, d_n(x, y)\}$$

   is a metric on $\Pi_{i=1}^n X_i$.