TOPOLOGY HW 4 • FALL 2021 • PROF. BOYLAND

This HW uses material from Lectures 18b-22.

1. Say that $(x, y) \sim (x', y')$ if and only if

$$x^2 + y^2 = (x')^2 + (y')^2$$

   (a) Show that this is an equivalence relation on $\mathbb{R}^2$.
   (b) Describe $\mathbb{R}^2/\sim$. You don't need to give a complete proof, just some justification for your answer.

2. Show that $\mathbb{R}^{\mathbb{Z}_+}$ with the uniform topology is not connected. Hint: consider bounded vs. unbounded sequences.

3. Is $\mathbb{R}_\ell$ (the real line with the lower limit topology) connected? Prove your result.

4. A chain of subsets is a collection A_n for $n \in \mathbb{Z}_+$ with the property that for all i, $A_i \cap A_{i+1} \neq \emptyset$. If each A_n in a chain is connected, show that $\cup_{n \in \mathbb{Z}_+} A_n$ is also connected.

5. In $\mathbb{R}$ with the finite complement topology, show that every infinite set is connected.

6. The unit circle is $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ with the subspace topology.

   (a) Show that S^1 is connected.
   (b) If $f : S^1 \to \mathbb{R}$ is continuous, show there exist some $(x, y) \in S^1$ so that $f(x, y) = f(-x, -y)$. ((x, y) and $(-x, -y)$ are called antipodal points)

7. If A is a countable set, show that $\mathbb{R}^2 - A$ is path connected. (Hint: How many lines are there coming out of a single point? You may use the fact that $\mathbb{R}$ is uncountable.)

8. Prove or disprove:

   (a) If A is connected, then $\mathrm{Bd}(A)$ and $\mathrm{Int}\, A$ are also.
   (b) If $\mathrm{Bd}(A)$ is connected, then A is connected.
   (c) If $\mathrm{Int}(A)$ is connected, then A is connected.

9. If X is locally path connected show that connected open sets are path connected.