TOPOLOGY REVIEW 1 • FALL 2021 • PROF. BOYLAND

1. Let Y be the set of all sequences in $\{0,1\}^{\mathbb{Z}_+}$ that are eventually zero. So $\underline{x} \in Y$ if and only if there exists an N so that $x_i = 0$ for $i \geq N$. Show that Y is countable.

2. Construct a homeomorphism from $\{0,1\}^{\mathbb{Z}_+}$ to a proper subset of itself where $\{0,1\}^{\mathbb{Z}_+}$ is given the product topology.

3. Prove or disprove.

   (a) $\overline{A \cap B} = \overline{A} \cap \overline{B}$

   (b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

   (c) $\overline{A - B} = \overline{A} - \overline{B}$

   (d) $\operatorname{Int}(A \times B) = \operatorname{Int} A \times \operatorname{Int} B$

   (e) $\operatorname{Int}(A \cup B) = \operatorname{Int} A \cup \operatorname{Int} B$

   (f) $\operatorname{Int}(A - B) = \operatorname{Int} A - \operatorname{Int} B$

4. Give $\mathbb{R}$ the finite complement topology.

   (a) To which point or points does the sequence $x_n = 1/n$ converge.

   (b) What is $\overline{\{2\}}$.

5. Assume $A \subset X$ and $f : A \to Y$ is continuous with Y Hausdorff. If there are continuous $g : \overline{A} \to Y$ and $h : \overline{A} \to Y$ that both extend f i.e. $f(a) = g(a) = h(a)$ for all $a \in A$ show that $f = g$. This says that continuous functions extend uniquely to the closure.

6. Let $B \subset \mathbb{R}^{\mathbb{Z}_+}$ be defined as

$$B = \prod_{i \in \mathbb{Z}_+} (-1,1) = (-1,1) \times (-1,1) \times (-1,1) \times \dots$$

   .

   (a) Is B open in the box topology?

   (b) Is B open in the uniform topology?

   (c) Is B open in the product topology?

7. If (X, d) is a metric space and $A \subset X$ for $x \in X$ define

$$d(x, A) = \inf\{d(x,a) \colon a \in A\}.$$

   (a) Show that the function $f : X \to [0, \infty)$ defined by $f(x) = d(x, A)$ is continuous

   (b) Show that $x \in \overline{A}$ if and only if $f(x) = 0$.

8. If (X, d) is a metric space show that for any x and $\epsilon > 0$,

$$\overline{B_\epsilon(x)} \subset \{y \in X \colon d(y,x) \leq \epsilon\}.$$

9. If $(X, \mathcal{T}_X)$ is a topological space and $f : X \to Y$ is a surjective function, is

$$\{f(U) \colon U \in \mathcal{T}_X\}$$

a topology on Y? Prove your answer.

10. Assume $A \subset X$.

    (a) Show that $\mathrm{Bd}(A)$ is a closed set

    (b) Prove that if $\mathrm{Bd}(A) = \emptyset$ then A is both open and closed.