TOPOLOGY REVIEW FOR EXAM 1 • SPRING 2022 • PROF. BOYLAND

This exam covers material from Lectures 1-16 and HW assignments 1-3. You should also review these HW for the exam.

1. Give the diagrammatic definition of the free product of a collection of groups $\{G_\lambda\}_{\lambda\in\Lambda}$ with respect to monomorphisms $\phi_\alpha : G_\alpha \to G$, state how it characterizes the free product and then prove this characterization.

2. State the Seifert-van Kampen Theorem and use it to carefully compute the fundamental group of the wedge of three circles.

3. Compute the fundamental group $S^2 \times B^2 \times S^1$

4. Show that X is simply connected if and only if and only if any two paths which start and end at the same point' are path homotopic.

5. (a) Give the definition of retract and deformation retract
   (b) Show that the retract of a Hausdorff space is Hausdorff.
   (c) Let $X = [-2, 2]^2 - \{0\}$ and let A be the square with corners $(1, 1), (-1, 1), (1, -1), (-1, -1)$. Show explicitly that A is a deformation retract of X.
   (d) Show that the point $x_0 = (1, 0)$ is a retract of X but not a deformation retract.

6. Define what it means for $p : E \to B$ to be a covering space and show that if B is simply connected then p is a homeomorphism.

7. Assume $U, V \subset X$ are open and simply connected, $X = U \cup V$, and $U \cap V \neq \emptyset$ and is path connected. Show that X is simply connected.

8. Assume that there is no retract $B^n \to S^{n-1}$ for all $n > 2$. Show that any continuous function $f : B^n \to B^n$ has a fixed point.

9. Assume X, U, V, x_0 satisfy the hypothesis of the SVK theorem and that V is simply connected. Show that there is an isomorphism

$$\pi_1(U, x_0)/N \to \pi_1(X, x_0)$$

   where N is the least normal subgroup of $\pi_1(U, x_0)$ that contains the image of the homomorphism

$$i_1 : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$$