TOPOLOGY REVIEW 2 • FALL 2021 • PROF. BOYLAND

1. Prove or disprove. Assume X is locally path connected and $A \subset X$ is connected, then A is path connected.

2. Assume X is second countable and A is an uncountable subset. Show that A has uncountably many limit points.

3. Assume $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and there is a constant $k > 1$ with $f'(x) > k$ for all $x \in \mathbb{R}$. Show that f has a unique fixed point x_0 and if $x > x_0$ then $f^n(x) \to \infty$ and if $x < x_0$ then $f^n(x) \to -\infty$. Hint: consider the functional inverse g with $f \circ g = \mathrm{id}$.

4. Assume that X is a Baire space and $X = \cup_{n\in\mathbb{Z}_+} B_n$, then for some n, $\mathrm{Int}(\overline{B_n}) \neq \emptyset$.

5. $A \subset X$, and $C \subset X$ is connected and intersects both A and $X - A$, then C also intersects intersects $\mathrm{Bd}(A)$

6. Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on X with $\mathcal{T} \subset \mathcal{T}'$. What does compactness in one of these topologies imply about compactness in the other.

7. Show that a connected metric space with more than one point is uncountable.

8. Show that a Lindelöf metric space is second countable

9. Prove the following: Let (Y, d) be a metric space. Let B be a bounded subset of Y. Then $\overline{B}$ is bounded and $\mathrm{diam}(B) = \mathrm{diam}(\overline{B})$.

10. Assume X, Y are topological space with Y a metric space and $\mathcal{C}(X, Y)$ is all continuous finctions from X to Y with the uniform topology. Show that the evaluation map $e : X \times \mathcal{C}(X, Y) \to Y$ given by $e(x, f) = f(x)$ is continuous

11. If A is compact in a metric space (X, d) then for any $x \in X$ there exists an $a \in A$ with $d(x, A) = d(x, a)$.

12. Let X be a set, let $f_n : X \to \mathbb{R}$ be a sequence of functions, and let $f : X \to \mathbb{R}$ be a function. Let $\overline{\rho}$ denote the uniform metric on the space $\mathbb{R}^X$. Prove that f_n converges uniformly to f if and only if the sequence f_n converges to f as elements of the metric space $(\mathbb{R}^X, \overline{\rho})$.