HW3 – Due Monday, October 17, start of class

No electronic submissions, only hard copy

1. (a) If $\{x_n\}$ is a sequence with $d(x_n, x_{n+1}) < 3^{-n}$ for all n, show that $\{x_n\}$ is Cauchy.

   (b) For each n let
$$x_n = \sum_{i=0}^{n} \frac{1}{3^i}.$$
   Show that x_n is a Cauchy sequence and compute its limit.

2. Use the proof of the contraction mapping theorem to show that if F is a contraction with constant r with fixed point p, then for any initial point z
$$d(F^n(z), p) \leq \frac{r^n d(z, F(z))}{1 - r}.$$
   Hint: You may use this fact without proof: Assume that $\{x_n\}$ is a Cauchy sequence that converges to x. Thus given ϵ there is an N so that $d(x_n, x_m) < \epsilon$ when $n, m > N$. It is then also the case that $d(x_n, x) \leq \epsilon$ when $n > N$.

3. Let
$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1/2 \\ 1/3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$
   (a) Show that F is a contraction mapping. What is its contraction constant?

   (b) Compute the unique fixed point p of F.

   (c) Let
$$z_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
   Compute $z_3 = F^3(z_0)$ and $d(z_3, p)$ and compare it to the estimate you got for question 2. You may use a calculator on this question.

4. For a sequence $\{x_n\}$, a subsequence $\{x_{n_i}\}$ is formed by picking a subset n_i of the indices n. For example if $x_n = 1/n$, then $1/2, 1/4, 1/6, \ldots$ is a subsequence.

   (a) If $\{x_n\}$ converges, show that any subsequence converges.

   (b) Prove or disprove the converse: If every subsequence of $\{x_n\}$ converges, then $\{x_n\}$ converges.

   (c) If $\{x_n\}$ is a Cauchy sequence, show that every subsequence is also a Cauchy sequence.

   (d) (extra credit, harder) If $\{x_n\}$ is a Cauchy sequence and it has one convergence subsequence, then $\{x_n\}$ also converges.